Math 8
Winter 2020
Taylor Polynomials and Taylor Series Day 4

Error Estimates

We know that the Maclaurin series for e^x is given as the limit of the Taylor polynomials,

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

and we have also learned that this series actually converges to e^x for every value of x,

$$e^x = \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

In particular, substituting $x = 1$, we get

$$e = e^1 = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

We can estimate e using Taylor polynomials, or partial sums of the Taylor series,

$$e \approx T_n(1) = \sum_{k=0}^{n} \frac{1}{k!}.$$

The error in this estimate is the difference between the estimate and the actual value,

$$error = |e - T_n(1)| = \left| e - \sum_{k=0}^{n} \frac{1}{k!} \right| = \left| \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} \right| = \left| \sum_{k=n+1}^{\infty} \frac{1}{k!} \right|.$$

In homework, you found the error in problems like this by calculating $|e - T_n(1)|$. However, this works only if you already know the actual value of e.

Today, we will learn a couple of ways of bounding the error

$$\left| \sum_{k=0}^{\infty} a_k - \sum_{k=0}^{n} a_k \right| = \left| \sum_{k=n+1}^{\infty} a_k \right|$$

in using a partial sum $S_n = \sum_{k=0}^{n} a_k$ to approximate the sum $\sum_{k=0}^{\infty} a_k$. Bounding the error means finding a number b such that the error is guaranteed to be smaller than b.

Alternating Series

Suppose the series $\sum_{k=0}^{\infty} a_k$ satisfies all three conditions of the alternating series test:
(1.) The terms a_n alternate between positive and negative.
(2.) The terms a_n are decreasing in absolute value, that is, $|a_{n+1}| \leq |a_n|$ for all n.
(3.) The terms a_n are approaching zero, $\lim_{n\to\infty} a_n = 0$.

The picture accompanying the proof of the alternating series test at the beginning of Section 11.5 of the textbook shows that the sum of the series must be between 0 and the first term a_0:

Proposition: If the series $\sum_{k=0}^{\infty} a_k$ satisfies all three conditions of the alternating series test:
(1.) The terms a_n alternate between positive and negative;
(2.) The terms a_n are decreasing in absolute value, that is, $|a_{n+1}| \leq |a_n|$ for all n;
(3.) The terms a_n are approaching zero, $\lim_{n\to\infty} a_n = 0$;
then

$$\left|\sum_{k=0}^{\infty} a_k\right| \leq |a_0|.$$

We can apply this to any tail end of the series as well:

$$\left|\sum_{k=n+1}^{\infty} a_k\right| \leq |a_{n+1}|.$$

We may call this the error bound from the alternating series test.

Example: Find n so that the approximation

$$e^{-1} \approx \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

is correct to within 2 decimal places (an error of at most .005).

We know from the fact that the Taylor series for e^x converges to e^x that

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

so the error we are looking for is

$$\left|e^{-1} - \sum_{k=0}^{n} \frac{(-1)^k}{k!}\right| = \left|\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} - \sum_{k=0}^{n} \frac{(-1)^k}{k!}\right| = \left|\sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}\right|.$$

Because the sum $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ satisfies the conditions of the alternating series test, we know

$$error = \left| \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \right| \leq \left| \frac{(-1)^{n+1}}{(n+1)!} \right| = \frac{1}{(n+1)!},$$

so we need to find n such that

$$\frac{1}{(n+1)!} \leq .005 = \frac{5}{1000} = \frac{1}{200},$$

or

$$(n+1)! \geq 200.$$

A table of factorials (or a little computation) will tell us that $5! = 120$, and $6! = 720$, so $n+1 = 6$, or $n = 5$, will do.

$$e^{-1} \approx 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} = \frac{11}{30}$$

and this approximation is correct to two decimal places.

Example: Find n so that the approximation

$$e^{-10} \approx \sum_{k=0}^{n} \frac{(-10)^k}{k!}$$

is correct to within 2 decimal places (an error of at most .005).

As before, we want to find n such that

$$error = \left| \sum_{k=n+1}^{\infty} \frac{(-10)^k}{k!} \right| \leq \frac{1}{200},$$

but now our series does not satisfy condition (2) of the alternating series test. The first few terms of the series are $1, -10, 50$, which are not decreasing in absolute value. The terms do begin to decrease in absolute value eventually.

In general, for this series we have $a_{n+1} = a_n \left(\frac{-10}{(n+1)!} \right)$. If $n \geq 10$ we have $\left| \frac{10}{(n+1)!} \right| < 1$, so for $n \geq 10$ we can use the error bound from the alternating series test. This means that we must find $n \geq 10$ such that $\left| \frac{(-10)^{(n+1)}}{(n+1)!} \right| \leq .005$. (It turns out that $n = 29$ will do.)

The Comparision Test

If the terms of our series are not alternating, we need another method. One such method comes from the comparison test. Remember, the comparison test says that if $a_k \le b_k$ for every k, and $\sum_{k=0}^{\infty} a_k = A$ and $\sum_{k=0}^{\infty} b_k = B$, then $A \le B$. Here is an example:

Example: Find n such that the approximation

$$e \approx \sum_{k=0}^{n} \frac{1}{k!}$$

is correct to within two decimal places.

As in our earlier problems, we want to find n such that we have

$$error = \sum_{k=n+1}^{\infty} \frac{1}{k!} \le .005 = \frac{1}{200}.$$

We will estimate the error by comparing the series to a series whose sum we know, a geometric series.

For our series, we have $a_k = \dfrac{1}{k!}$ so that $a_{k+1} = a_k \left(\dfrac{1}{k+1}\right)$. If we are looking at the tail end that gives our error,

$$\sum_{k=n+1}^{\infty} \frac{1}{k!},$$

we have $k+1 > n+1$ for all k, so $a_{k+1} = a_k \left(\dfrac{1}{k+1}\right) < a_k \left(\dfrac{1}{n+1}\right)$. This tells us that, term-by-term, our tail end series is less than or equal to the series

$$a_{n+1},\ a_{n+1}\left(\frac{1}{n+1}\right),\ a_{n+1}\left(\frac{1}{n+1}\right)^2,\ a_{n+1}\left(\frac{1}{n+1}\right)^3, \ldots$$

This is a geometric series with first term $a_{n+1} = \dfrac{1}{(n+1)!}$ and ratio $\dfrac{1}{(n+1)}$, so its sum is $\dfrac{\frac{1}{(n+1)!}}{1 - \frac{1}{(n+1)}} = \dfrac{1}{n!(n)}$. In this case we want to make this less than $\dfrac{1}{200}$, or $n!(n) > 200$. From a table of factorials we can find $4!(4) = 24(4) = 96$, and $5!(5) = 120(5) = 600$, so $n = 5$ will do.

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2\,\frac{43}{60}$$

and this approximation is correct to two decimal places.