Math 8
Winter 2020
Taylor Polynomials and Taylor Series Day 5

Here is one more convergence test.

Proposition: (the ratio test for nonnegative series) If $a_n \geq 0$ for all n, and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L,$$

then

$$\sum_{n=0}^{\infty} a_n \text{ is } \begin{cases} \text{convergent} & \text{if } L < 1; \\ \text{divergent} & \text{if } L > 1; \\ \text{we cannot tell from this test} & \text{if } L = 1. \end{cases}$$

We can rephrase this using our earlier proposition that absolutely convergent sequences are always convergent.

Proposition: (the ratio test) For any series, if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

then

$$\sum_{n=0}^{\infty} a_n \text{ is } \begin{cases} \text{absolutely convergent} & \text{if } L < 1; \\ \text{divergent} & \text{if } L > 1; \\ \text{we cannot tell from this test} & \text{if } L = 1. \end{cases}$$

There is a proof of the ratio test in the last section. Intuitively, if we have a nonnegative series with $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$, then a tail end of the series behaves very much like a geometric series with ratio $\frac{1}{2}$, which converges. On the other hand, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$, then eventually the terms of the series are getting larger and larger, and the sum of larger and larger numbers must approach $+\infty$.

The ratio test will be useful for Taylor series.

Radius of Convergence

Taylor series centered at $x = a$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

are examples of power series centered at $x = a$

$$\sum_{k=0}^{\infty} c_k(x-a)^k \qquad \text{(each } c_k \text{ is a constant).}$$

We can use a power series to define a function,

$$g(x) = \sum_{k=0}^{\infty} c_k(x-a)^k,$$

whose domain is the set of x for which the power series converges.

We hope that the Taylor series for $f(x)$ not only converges, but converges to $f(x)$.

To find the set of x for which a given power series converges, a good place to start is the ratio test. For example, consider

$$\sum_{k=0}^{\infty} k(x-1)^k.$$

To see whether this series converges for a particular value of x, we use the ratio test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)(x-1)^{k+1}}{k(x-1)^k}\right| = \lim_{k\to\infty}\left|\frac{k+1}{k}(x-1)\right| =$$

$$\left(\lim_{k\to\infty}\left|\frac{k+1}{k}\right|\right)|x-1| = |x-1|.$$

By the ratio test, this power series converges absolutely if $|x-1| < 1$ and diverges if $|x-1| > 1$. The ratio test doesn't tell us what happens for $|x-1| = 1$. Therefore the ratio test tells us that the domain of the function defined by this power series contains $(0, 2)$, and may or may not contain the points 0 and 2.

To see whether it contains the endpoints of the interval, we can plug them in and see what happens. For $x = 2$, we have

$$\sum_{k=0}^{\infty} k(x-1)^k = \sum_{k=0}^{\infty} k(1)^k = 0 + 1 + 2 + 3 + \cdots$$

and for $x = 0$ we have

$$\sum_{k=0}^{\infty} k(x-1)^k = \sum_{k=0}^{\infty} k(-1)^k = 0 - 1 + 2 - 3 + \cdots$$

both of which diverge. Therefore the domain of this function is $(0, 2)$.

In general, we can apply the ratio test to the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ and get

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{c_{k+1}(x-a)^{k+1}}{c_k(x-a)^k}\right| = \lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}(x-a)\right| = \left(\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right|\right)|x-a|.$$

If $\left(\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right|\right) = Q$, then we get

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = Q|x-a|,$$

and the power series converges absolutely when

$$Q|x-a| < 1, \text{ or } |x-a| < \frac{1}{Q},$$

and diverges if

$$Q|x-a| > 1, \text{ or } |x-a| > \frac{1}{Q}.$$

We call $R = \frac{1}{Q}$ the *radius of convergence.* (If $Q = 0$ the radius of convergence is $+\infty$, and if $Q = +\infty$ the radius of convergence is 0.) The power series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$. For $|x-a| = R$, it may converge or diverge, depending on the power series.

It turns out that power series always behave this way, even if $\left(\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right|\right)$ does not converge.

Proposition 0.1. *A power series* $\sum_{k=0}^{\infty} c_k(x-a)^k$ *always has a radius of convergence* R *with* $0 \leq R \leq \infty$. *The power series converges absolutely for* $|x-a| < R$ *and diverges for* $|x-a| > R$.

If $R = 0$ *the power series converges only for* $x = a$ *and if* $R = \infty$ *it converges for all* x. *For* $0 < R < \infty$, *the set of* x *for which the power series converges is one of the intervals*

$$(a-R, a+R),\ [a-R, a+R),\ (a-R, a+R],\ \text{or } [a-R, a+R].$$

This is called the interval of convergence of the series.

New Taylor Series from Old

If we compute the Taylor series for $f(x)$ centered at a directly from the formula for Taylor series, we can use the ratio test to find the radius of convergence. For x within that radius of convergence of a, we hope that the Taylor series for $f(x)$ not only converges, but converges to $f(x)$. The ratio test cannot tell us that.

However, there is another way to arrive at Taylor series, and know they converge to the function.

Theorem: Suppose the function $f(x)$ is defined by a power series centered at a with radius of convergence R,

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k.$$

Then the term-by-term derivative of the power series has the same radius of convergence, and gives the derivative of $f(x)$:

$$f'(x) = \sum_{k=0}^{\infty} kc_k(x-a)^{k-1}.$$

The same is true for the term-by-term indefinite integral,

$$\int f(x)\,dx = C + \sum_{k=0}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1},$$

and definite integral, as long as b and d are in $(a-R, a+R)$,

$$\int_b^d f(x)\,dx = \left[\sum_{k=0}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1}\right]\Bigg|_{x=b}^{x=d} = \sum_{k=0}^{\infty}\left(\left[c_k \frac{(x-a)^{k+1}}{k+1}\right]\Bigg|_{x=b}^{x=d}\right).$$

In particular, for $a-R < x < a+R$ (inside the radius of convergence)

$$\int_a^x f(u)\,du = \left[\sum_{k=0}^{\infty} c_k \frac{(u-a)^{k+1}}{k+1}\right]\Bigg|_{u=a}^{u=x} = \sum_{k=0}^{\infty}\left(\left[c_k \frac{(x-a)^{k+1}}{k+1}\right]\right).$$

Example: Use this theorem to find a power series expansion (Taylor series) for $\ln(x+1)$ centered at $a=0$.

For $|x| < 1$ we have:

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty}(-x)^k = \sum_{k=0}^{\infty}(-1)^k x^k$$

$$\int \frac{1}{1+x}\,dx = C + \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1} = C + \sum_{i=1}^{\infty}(-1)^{i-1}\frac{x^i}{i}$$

Evaluating the integral,

$$\ln(x+1) = C + \sum_{i=1}^{\infty}(-1)^{i+1}\frac{x^i}{i} = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Plugging in $x=0$,

$$\ln(1) = C \qquad \text{therefore} \qquad C = 0.$$

$$\ln(x+1) = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{x^i}{i} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Our theorem says this holds for $|x| < 1$, and the series diverges for $|x| > 1$. The theorem doesn't tell us what happens if $|x| = 1$. If we care, we have to check that separately.

For this example, for $x = 1$ the series does converge to $\ln(2)$. For $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$. This is -1 times the harmonic series, and the harmonic series diverges to ∞.

Example: It is possible to show that for all x (in other words, with radius of convergence $R = \infty$),

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{(2k+1}}{(2k+1)!} \right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

By this theorem we can take derivatives of each side, differentiating the power series term-by-term, to get

$$\cos(x) = \sum_{k=0}^{\infty} (2k+1)(-1)^k \left(\frac{x^{2k}}{(2k+1)!} \right) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{2k}}{(2k)!} \right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ,$$

with the same radius of convergence $R = \infty$. This shows the Maclaurin series for $\cos(x)$ also converges to $\cos(x)$ for every x.

Example: We have seen that for $|x| < 1$ we have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

Substituting $x = -u^2$, we get

$$\frac{1}{1+u^2} = \sum_{k=0}^{\infty} (-u^2)^k = \sum_{k=0}^{\infty} (-1)^k u^{2k}.$$

This holds for $| - u^2| < 1$, which is to say, $|u| < 1$. Applying our theorem, we get

$$\int_0^x \frac{1}{1+u^2}\, du = \int_0^x \left(\sum_{k=0}^{\infty} (-1)^k u^{2k} \right) dx = \sum_{k=0}^{\infty} \left(\int_0^x (-1)^k u^{2k}\, du \right);$$

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

By the theorem, this has the same radius of convergence, namely 1, so this is true for $|x| < 1$.

The theorem doesn't tell us whether this is true for $|x| = 1$, that is, for $x = 1$ or for $x = -1$. It turns out that it is true in both cases, so the interval of convergence for this series is $[-1, 1]$, and it converges to $\arctan(x)$ for every x in that interval.

Proof That the Ratio Test Works: Suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

First, suppose that $L > 1$, and choose $\varepsilon > 0$ small enough that $L - \varepsilon > 1$. By the definition of limit, there is some N such that whenever $n > N$ we have $\left| \frac{a_{n+1}}{a_n} \right| - L < \varepsilon$. This means that

$$\left| \frac{a_{n+1}}{a_n} \right| > L - \varepsilon > 1,$$

and that means that $|a_{n+1}| > |a_n|$. Since the individual terms of the series are getting larger in absolute value, they are not approaching zero, and so the series must diverge.

Now, suppose that $L < 1$, and choose $\varepsilon > 0$ small enough that $L + \varepsilon < 1$. By the definition of limit, there is some N such that whenever $n > N$ we have $\left| \frac{a_{n+1}}{a_n} \right| - L < \varepsilon$. This means that

$$\left| \frac{a_{n+1}}{a_n} \right| < L + \varepsilon.$$

Consider the series,

$$\sum_{n=N+1}^{\infty} |a_n|.$$

For this series, we have

$$\frac{|a_{n+1}|}{|a_n|} < L + \varepsilon,$$

$$|a_{n+1}| \leq (L + \varepsilon)|a_n|,$$

which means the terms of the series are less than or equal to the terms of a geometric series with first term a_{N+1} and ratio $L+\varepsilon$. Since $L+\varepsilon<1$, this geometric series converges, and therefore the smaller series $\displaystyle\sum_{n=N+1}^{\infty}|a_{n+1}|$ also converges.

Since a tail end of the series $\displaystyle\sum_{n=0}^{\infty}|a_n|$ converges, the series itself converges. This means the original series $\displaystyle\sum_{n=0}^{\infty}a_n$ is absolutely convergent, and therefore it is convergent.