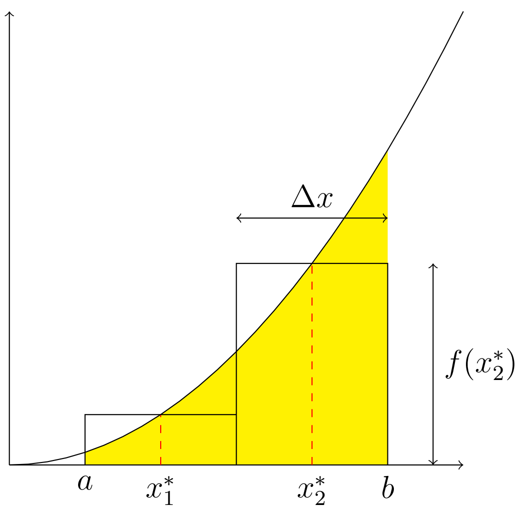
Math 8
Winter 2020
Applications of Integration Day 6

Riemann Sums

You may remember that the definite integral was originally defined by approximating the area under a curve by a sum of areas of rectangles:

We want to find $\int_a^b f(x)\,dx$, the area of the region above the x-axis and below the curve $y = f(x)$ for $a \leq x \leq b$.

First, we approximate the area as follows: Divide the interval $[a, b]$ into n-many small intervals of length $\Delta x = \dfrac{b-a}{n}$. Choose an x-value in each interval. Let x_i^* name the x-value chosen in interval number i. Approximate the area of the region above the i^{th} subinterval and below $y = f(x)$ by the area of a rectangle of height $f(x_i^*)$ and width Δx.



Area of i^{th} rectangle is $A_i \approx f(x_i^*)\Delta x$.

Approximate the area of the entire region as the sum of these areas of rectangles. This sum is called a Riemann sum for $f(x)$.

$$area = \sum_{i=1}^{n} A_i \approx \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Now find the actual area by taking a limit as $n \to \infty$, or as $\Delta x \to 0$:

$$area = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx.$$

Applying Riemann Sums

We can use this formula

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

to find applications of integration; that is, to figure out how to use integrals to calculate the answer to some problem.

Here is the basic idea: You want to find the value of a quantity Q that arises from some problem. Approximate Q by breaking the problem up into small pieces, approximating the quantity for each piece, and then adding up the answers. If the limit of your approximations, as the number of small pieces approaches infinity, is the correct answer Q, and if your approximation has the form of a Riemann sum, then you can express Q as a definite integral.

The next sections give some examples of this process. You may see any of these applications in Math 8, but you should pay attention to more than the specific formulas. You should also understand how these applications are developed. You can count on seeing at least one exam problem that asks you to evaluate some quantity using an integral, by first approximating the quantity you are interested in by a sum and then taking a limit. There are several examples of this process here.

The Net Change Theorem (for functions of time): This is just the Fundamental Theorem of Calculus, applied. The Fundamental Theorem of Calculus tells us that, if F has a continuous derivative on the interval $[a, b]$, then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

If t represents time, and $F(t)$ represents some quantity that changes over time, then $F'(t)$ represents the rate of change of $F(t)$ (with respect to time), and $F(b) - F(a)$ represents the *net change* in $F(t)$ between times $t = a$ and $t = b$.

("Net change" means that we allow decreases and increases to cancel each other out. For example, if the temperature is -15 degrees at 7 AM, rises to 13 degrees at 2 PM, and then drops to 11 degrees at 7 PM, then between 7 AM and 7 PM the temperature rises by 28 degrees and then falls by 2 degrees, and the net change is 26 degrees; a positive net change denotes an increase.[1])

This is what the Net Change Theorem says. Quoting directly from Stewart's calculus textbook[2]:

Net Change Theorem. The integral of a rate of change is the net change:

$$\boxed{\int_a^b F'(x)\ dx = F(b) - F(a)}.$$

[1] I got these numbers from the Weather Underground forecast for January 8, 2015 in Hanover, New Hampshire; degrees are in Fahrenheit.

[2] Stewart, James. *Calculus*, seventh edition. Cengage Learning, 2012.

In particular, the Net Change Theorem tells us that the integral of velocity is distance: For an object traveling along a straight path (with positive and negative directions chosen), if $F(t)$ denotes the distance from the starting point at time t, then $F'(t)$ denotes velocity, the rate of change of distance with respect to time, and the integral of the velocity between times $t = a$ and $t = b$

$$\int_a^b F'(t)\,dt$$

denotes the net distance traveled between times $t = a$ and $t = b$.

Again, for net distance, we allow distances in the positive and negative directions to cancel each other out. If you start at the corner of the green, walk a block south on Main Street, then walk a block north back to your starting point, you have walked 2 blocks, but traveled a net distance of zero.

Average Value: The average value of a function $f(x)$ for x in the interval $[a, b]$ is given by

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}.$$

This is very like the formula for computing the average of finitely many numbers: To find the average of a finite set of numbers, add up the numbers, and divide by the size of the set. To find the average value of a function $f(x)$ on an interval, use an integral to "total up" the values of f, and divide by the length of the interval.

Here is how we find this formula: To approximate the average value of $f(x)$ on the interval $[a, b]$, divide the interval $[a, b]$ into a large number of equal subintervals (say n subintervals), choose a point x_i^* in the i^{th} subinterval for $i = 1, 2, \ldots n$, and take the average of the values $f(x_i^*)$. This gives us

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i^*) = \frac{1}{b-a}\frac{b-a}{n}\sum_{i=1}^{n} f(x_i^*) = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\frac{b-a}{n} =$$

$$\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)^*\Delta x,$$

where $\Delta x = \dfrac{b-a}{n}$ is the size of the subintervals. This sum is just the Riemann sum that we used to approximate the area under a curve. When we compute the average as the limit of closer and closer approximations as $n \to \infty$, we get

$$\left(\frac{1}{b-a}\right)\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Note: Formal computations, such as the derivation of the formula for average value above, are of interest not only to theoretical mathematicians. If you are using calculus in engineering or economics or medical research, you don't want to be limited to the applications of integration on some list; you want to know how to recognize a new application of integration "in the wild," so to speak. You recognize a potential application of integration by seeing that you can approximate the thing you're interested in as a sum of small pieces, and get a better approximation by using a larger number of smaller pieces. Then, some computation like this is needed to tell you exactly what integral you should use to compute that thing you're interested in.

In this course, we will sometimes expect you to derive these formulas, such as this formula for average value, not just to apply them. In a physics or engineering course, you will often have to do problems, such as some of the computations of work in the section on work in our textbook, that essentially require you to figure out the appropriate integral in this way.

Volumes by Slicing: Suppose we want to find the volume of a solid object occupying the region in space between $x = a$ and $x = b$, where x is measured along some straight axis.

We can approximate the volume of the object by cutting it, perpendicular[3] to the x-axis, into many thin slices of thickness Δx. The volume of each thin slice can be approximated by the cross-sectional area $A(x_i^*)$ at some particular point x_i^* on the ais, times the thickness Δx of the slice. (Here we are setting $A(x)$ to be the area of the slice we get at point x, or the cross-sectional area at x.) Then we can approximate the volume of the object by adding up these approximate volumes $A(x_i^*)\Delta x$ of the slices,

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x,$$

and find the exact volume by taking the limit as the number of slices approaches infinity

$$V = \lim_{n\to\infty} \left(\sum_{i=1}^{n} A(x_i^*)\Delta x \right).$$

We recognize this as the limit of Riemann sums, and therefore we know we can say,

$$V = \lim_{n\to\infty} \left(\sum_{i=1}^{n} A(x_i^*)\Delta x \right) = \int_a^b A(x)\,dx.$$

We have shown we can find the volume of our object by integrating the object's cross-sectional area between $x = a$ and $x = b$. More precisely, suppose $A(x)$ is the cross-sectional area at x; that is, the cross-sectional area we get if we slice the object at the point x, perpendicularly to the axis. Then the volume of the object is given by

$$\boxed{V = \int_a^b A(x)\,dx}.$$

We can also give an informal argument that this makes sense. If the cross-sectional area is the same at every point, the volume of the object is the product of its length and its

[3]It is important that our slices are perpendicular to the axis.

cross-sectional area. For example, a cylinder of length h and cross-section a circle of radius r has volume $V = \pi r^2 h$, a formula that may be familiar. It makes sense that if the cross-sectional area varies, you could possibly get the volume by multiplying the length by the average cross-sectional area. Using the formula for average value, we get

$$V = (b - a)\,\frac{1}{b - a}\int_a^b A(x)\,dx = \int_a^b A(x)\,dx.$$

Warning: Informal arguments like this can sometimes lead us astray; that is why we need precise mathematical arguments. Our intuition can mislead us — in real life, all too many things that make sense turn out not to be true — and therefore the reasoning with approximations and limits is not only more formal but also more reliable.

For example, you might want to check that if you try to calculate the volume of the solid obtained by revolving the function $y = f(x)$ for $a \le x \le b$ around the x-axis by taking the average value C of $f(x)$ on that interval, and then finding the volume of the solid obtained by revolving the constant function $g(x) = C$ for $a \le x \le b$ around the x-axis, you will get the wrong answer. (Try it in the simple case $f(x) = x$, $a = 0$, $b = 1$.)