**Math 8**
**Winter 2020**
**Volumes of Revolution**

One of the applications of integration we derived in the last notes was a method of finding volumes:

**Volumes by Slicing**: To find the volume of a solid object occupying the region in space between $x = a$ and $x = b$, where $x$ is measured along some straight axis, integrate the object's cross-sectional area between $x = a$ and $x = b$. More precisely, suppose $A(x)$ is the cross-sectional area at $x$; that is, the cross-sectional area we get if we slice the object at the point $x$, perpendicularly to the axis. Then the volume of the object is given by

$$\boxed{V = \int_a^b A(x)\,dx}.$$

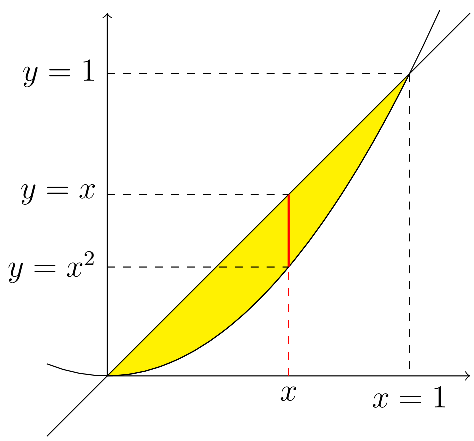
**Volumes of Revolution by Slicing**: This is a special case of volumes by slicing. Suppose the portion of the graph $y = f(x)$ for $a \leq x \leq b$ is revolved around the $x$-axis, and we want to find the volume of the solid region inside the surface it sweeps out. (Imagine taking the half circle $y = \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$ and revolving it around the $x$-axis. The resulting surface is a sphere centered at the origin of the $xy$-plane. We want to find the volume inside that sphere.)

Consider slicing this solid region perpendicular to the $x$-axis at a point $x$. The resulting cross-section is a disc with radius $f(x)$ (or $|f(x)|$ if $f(x) < 0$) and so the cross-sectional area at $x$ is $A(x) = \pi(f(x))^2$. Therefore, the volume is

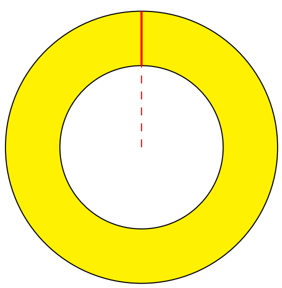
$$V = \int_a^b A(x)\,dx = \int_a^b \pi(f(x))^2\,dx.$$

**Note:** This method is also called volumes by discs, as the cross-section at $x$ is a disc. A slightly more general method is called volumes by washers, as the cross-section at $x$ looks like the piece of hardware called a washer, as in the following example.

**Example:** Find the volume of the region generated by revolving the region above the parabola $y = x^2$ and below the line $y = x$ around the $x$-axis.




The yellow region above is revolved around the $x$-axis. To visualize the cross-section of the resulting solid at $x$, imagine revolving the solid red line segment around the $x$-axis. You will get a two-dimensional region that looks like a disc with a smaller disc removed:



The radius of the smaller disc is the length of the dashed red line, or $x^2$, and the radius of the larger disc is the length of the dashed red line together with the solid red line, or $x$. Therefore the cross-sectional area is the area of the larger disc minus the area of the smaller disc,

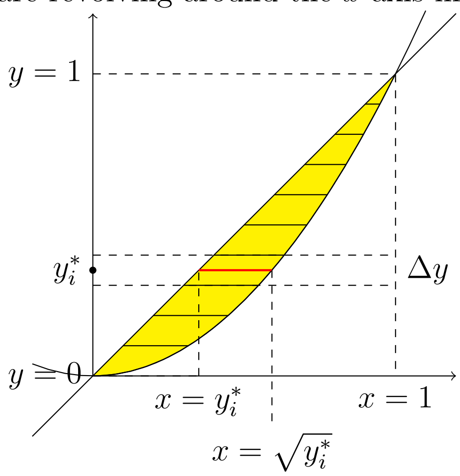
$$A(x) = \pi x^2 - \pi(x^2)^2 = \pi(x^2 - x^4),$$

and we can integrate this to get the volume:

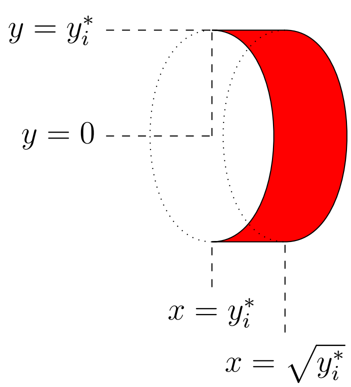
$$\int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx = \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_{x=0}^{x=1} = \frac{2\pi}{15}.$$

**Volumes of Revolution by Shells:**

Here is a different method, not volumes by slicing, to find a volume of revolution. We'll look at the same example as before, and divide the region we are revolving around the $x$-axis into thin horizontal bands of height $\Delta y$:




We focus on the band containing the red line segment at height $y_i^*$. This red line segment has left endpoint at $x = y_i^*$ and right end at $x = \sqrt{y_i^*}$, so its length is $\sqrt{y_i^*} - y_i^*$. If we revolve that line segment around the $x$-axis, we get a cylinder, of height (or length, since it is oriented horizontally instead of vertically) $\sqrt{y_i^*} - y_i^*$ and radius $y_i^*$, as in the picture below.




This cylinder is a surface with area (via the formula $A = 2\pi rh$) equal to $2\pi(y_i^*)(\sqrt{y_i^*} - y_i^*)$.

If $\Delta y$ is small enough, when we revolve the band containing that red line segment around the $x$-axis, we get a three-dimensional object that looks approximately like that same cylindrical surface, made three-dimensional by giving it a thickness of $\Delta y$. The volume of that object is approximately $\Delta y$ times the surface area of the cylinder, or $2\pi(y_i^*)(\sqrt{y_i^*} - y_i^*)\Delta y$.

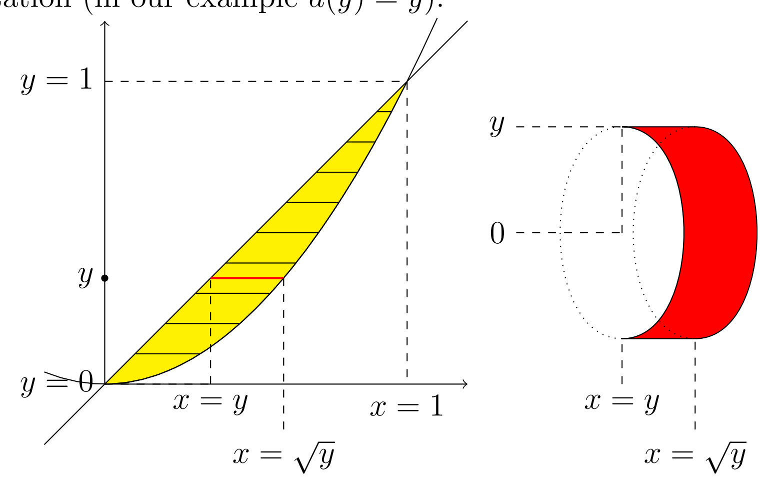
We approximate the volume of our object by adding up the approximate volumes of all the slices,

$$V \approx \sum_{i=1}^{n} 2\pi(y_i^*)(\sqrt{y_i^*} - y_i^*)\Delta y,$$

and take the limit as $n \to \infty$ to find an integral that gives the volume,

$$V = \int_0^1 2\pi(y)(\sqrt{y} - y)dy.$$

Here is the method, in general. We are revolving our region around one axis (the $x$-axis in this example) and we are going to integrate along a second axis perpendicular to the axis of rotation (the $y$-axis). At each point $y$ on the axis of integration, we draw a line segment across our region, of length $\ell(y)$ (in our example, $\ell(y) = (\sqrt{y} - y)$) and a distance $d(y)$ from the axis of rotation (in our example $d(y) = y$).

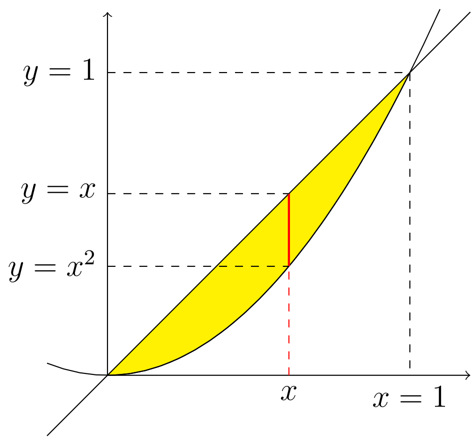


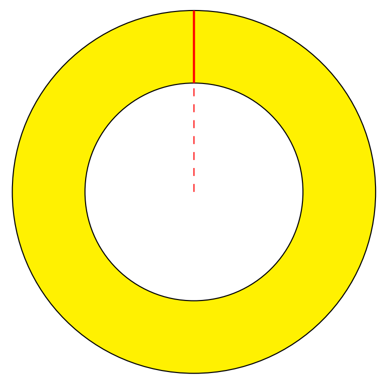

Revolving that line segment around the axis of rotation gives a cylinder of area $C(y) = 2\pi d(y)\ell(y)$. We integrate this cylindrical area over the interval $c \le y \le d$ our region occupies,

$$\int_c^d C(y)\,dy = \int_c^d 2\pi d(y)\ell(y)\,dy.$$

It is important to keep your axes straight, so let us recap:

Volumes by discs or washers: Integrate the cross-sectional area along the axis around which you are revolving. In this picture, we revolve around the $x$-axis, and we get $\int_a^b A(x)\,dx$.



Volumes by shells: Integrate the cylindrical-sectional area along an axis perpendicular to the one around which you are revolving. In this picture, we revolve around the $x$-axis, and we get $\int_c^d C(y)\,dy$.