Math 8
Winter 2020
Section 1
January 10, 2020

Differentiating e^x

Question: How do we show
$$\frac{d}{dx}e^x = e^x?$$

First, it depends on how e is defined. You may have seen
$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$
given as the definition of e. This isn't our textbook's approach, but let's use this definition, and try to compute the derivative of $f(x) = e^x$.
$$\frac{d}{dx}e^x = \lim_{t\to 0}\frac{e^{x+t}-e^x}{t} = \lim_{t\to 0}\frac{e^x(e^t-1)}{t} = e^x\lim_{t\to 0}\frac{e^t-1}{t}.$$

In other words, the derivative of e^x is a constant multiple of e^x, and that constant is $\lim_{t\to 0}\frac{e^t-1}{t}$. (We are taking it on faith that this limit exists.) We will call this limit L:
$$L = \lim_{t\to 0}\frac{e^t-1}{t} \qquad \frac{d}{dx}e^x = Le^x.$$
We want to show that $L = 1$.

The natural logarithm $\ln(x)$ is defined as the inverse function to e^x, so we can use the inverse function rule to compute its derivative:
$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \qquad \frac{d}{dx}(\ln(x)) = \frac{1}{Le^{\ln(x)}} = \frac{1}{Lx}.$$

Now we will use our definition of e:
$$e^1 = e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}e^{\ln\left(\left(1+\frac{1}{n}\right)^n\right)} = e^{\lim_{n\to\infty}\ln\left(\left(1+\frac{1}{n}\right)^n\right)}$$

This gives us (using l'Hôpital's Rule at step $*$)
$$1 = \lim_{n\to\infty}\ln\left(\left(1+\frac{1}{n}\right)^n\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} =^* \lim_{n\to\infty}\frac{\frac{1}{L\left(1+\frac{1}{n}\right)}\left(\frac{-1}{n^2}\right)}{\frac{-1}{n^2}} = \lim_{n\to\infty}\frac{\frac{1}{L\left(1+\frac{1}{n}\right)}}{1} = \frac{1}{L}.$$

So $L = 1$, which is what we needed to show.