Math 8
Winter 2020
Sequence Rules

You are free to use these rules in any homework or exam problem (unless the instructions say otherwise, such as, "use the definition of limit.") You do not have to cite the rule by name, as long as your reasoning is clear. For example, it is perfectly fine to say

$$\lim_{n\to\infty}\left(\frac{1}{1-x}-\frac{x^{n+1}}{1-x}\right)=\lim_{n\to\infty}\left(\frac{1}{1-x}\right)-\lim_{n\to\infty}\left(\frac{x^{n+1}}{1-x}\right).$$

Sequence Rules

1. (constant sequence rule)

   If $(a_n)_{n=0}^{\infty}$ is the constant sequence with value c (that is, $a_n = c$ for every n), then

   $$\lim_{n\to\infty} a_n = c.$$

2. (constant multiple rule)

   If c is a constant, then

   $$\left(\lim_{n\to\infty} a_n = A\right) \implies \lim_{n\to\infty}(ca_n) = cA.$$

3. (addition and subtraction rules)

   $$\left(\lim_{n\to\infty} a_n = A \;\&\; \lim_{n\to\infty} b_n = B\right) \implies \lim_{n\to\infty}(a_n \pm b_n) = A \pm B.$$

4. (multiplication rule)

   $$\left(\lim_{n\to\infty} a_n = A \;\&\; \lim_{n\to\infty} b_n = B\right) \implies \lim_{n\to\infty}(a_n b_n) = AB.$$

5. (division rule)

   If $b_n \neq 0$ for all n and $B \neq 0$, then

   $$\left(\lim_{n\to\infty} a_n = A \;\&\; \lim_{n\to\infty} b_n = B\right) \implies \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B}.$$

6. (continuous function rule)

   If f is continuous at A, then

   $$\left(\lim_{n\to\infty} a_n = A\right) \implies \lim_{n\to\infty}(f(a_n)) = f(A).$$

   For example, since $\lim_{n\to\infty}\frac{1}{n} = 0$, we know $\lim_{n\to\infty}\cos\left(\frac{1}{n}\right) = \cos(0) = 1$.

7. (horizontal asymptote rule)

$$\left(\lim_{x\to\infty} f(x) = A\right) \implies \lim_{n\to\infty} (f(n)) = A.$$

8. (limit comparison)

If $a_n \le b_n$ for all n, then

$$\left(\lim_{n\to\infty} a_n = A \;\&\; \lim_{n\to\infty} b_n = B\right) \implies A \le B.$$

9. (squeeze theorem)

If $a_n \le c_n \le b_n$ for all n, then

$$\left(\lim_{n\to\infty} a_n = A \;\&\; \lim_{n\to\infty} b_n = A\right) \implies \lim_{n\to\infty} (c_n) = A.$$

10. (tail end rule)

The sequences $(a_n)_{n=0}^{\infty}$ and $(a_n)_{n=k}^{\infty}$ have the same limit.

11. (decreasing differences rule)

If $(a_n)_{n=0}^{\infty}$ converges, then $\lim_{n\to\infty} (a_{n+1} - a_n) = 0$.

The converse of this is false, as you can see from the sequence

$$1,\ 1\frac{1}{2},\ 2,\ 2\frac{1}{3},\ 2\frac{2}{3},\ 3,\ 3\frac{1}{4}, \ldots,$$

which does not converge even though the differences of successive terms do approach zero.

We usually use this rule to show divergence: If $\lim_{n\to\infty} (a_{n+1} - a_n) \neq 0$ then $(a_n)_{n=0}^{\infty}$ does not converge.

12. (subsequence rule)

If $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$ (that means it is the same sequence but with some — possibly infinitely many — terms left out), then

$$\left(\lim_{n\to\infty} a_n = A\right) \implies \lim_{n\to\infty} (b_n) = A.$$

On the other hand, the original sequence may diverge even if the subsequence converges. For example,

$$1, 1, 1, 1, 1, \ldots$$

is a subsequence of

$$1, 2, 1, 3, 1, 4, 1, 5, \ldots$$

13. (monotone sequence theorem)

An increasing sequence must either converge to a limit or approach $+\infty$, and a decreasing sequence must either converge to a limit or approach $-\infty$.