Math 8
Winter 2020
Some Rules for Series

There is an extensive theory of sequences and series, most of which we will not see in Math 8. In this section, we state a few rules that should make sense. This is not a collection of facts to memorize. This is reassurance that your common sense conclusions about series and sequences are generally valid.

Generally the limits A and B in these rules are assumed to be numbers. The rules also apply to limits of ∞ and $-\infty$, as long as the expression you are evaluating is defined ($\infty + \infty = \infty$) rather than undefined ($\infty - \infty$ is undefined). Be warned that the quotient $\frac{\infty}{0}$ is undefined, *not* ∞. That is because if a_n approaches ∞ and b_n approaches 0 while oscillating between positive and negative values, then $\frac{a_n}{b_n}$ will also oscillate between positive and negative values, and therefore will not approach ∞.

You are free to use these rules in any homework or exam problem (unless the instructions say otherwise, such as, "use the definition of limit.") You do not have to cite the rule by name, as long as you make clear what fact you are using.

1. (constant multiple rule)

   If c is a constant, then

$$\left(\sum_{n=0}^{\infty} a_n = A\right) \implies \sum_{n=0}^{\infty}(ca_n) = cA.$$

2. (addition and subtraction rules)

$$\left(\sum_{n=0}^{\infty} a_n = A \ \& \ \sum_{n=0}^{\infty} b_n = B\right) \implies \sum_{n=0}^{\infty}(a_n \pm b_n) = A \pm B.$$

3. (tail end rule)

$$\sum_{n=0}^{\infty} a_n \text{ converges} \iff \sum_{n=k}^{\infty} a_n \text{ converges.}$$

   In fact,

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n.$$

4. (comparison rule)

   If $a_n \leq b_n$ for all n, then

$$\left(\sum_{n=0}^{\infty} a_n = A \ \& \ \sum_{n=0}^{\infty} b_n = B\right) \implies A \leq B.$$

5. (decreasing terms rule)

   If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{n\to\infty}(a_k) = 0$.

   The converse of this is false, as you can see from the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots,$$

   which does not converge even though the individual terms do approach zero.

6. (nonnegative series rule)

   If $a_n \geq 0$ for all n, then $\sum_{n=0}^{\infty} a_n$ either converges to a finite sum or approaches $+\infty$.

These rules basically follow from applying sequence rules to the sequences of partial sums. For example, the nonnegative series rule follows from the monotone sequence theorem, since if $a_n \geq 0$ for all n, then the sequence of partial sums is an increasing sequence, which must either converge to a number or diverge to $+\infty$.

Convergence

Proposition (the comparison test): Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are non-negative series. If $0 \leq b_n \leq a_n$ for all n, then

$$\sum_{n=0}^{\infty} a_n \text{ converges} \implies \sum_{n=0}^{\infty} b_n \text{ converges.}$$

Proposition: If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Proposition (the alternating series test): If a series $\sum_{k=0}^{\infty} a_k$ satisfies the following three conditions, then it converges:

(1.) The terms a_n alternate between positive and negative.
(2.) The terms a_n are decreasing in absolute value, that is, $|a_{n+1}| \leq |a_n|$ for all n.
(3.) The terms a_n are approaching zero, $\lim_{n\to\infty} a_n = 0$.