Math 8
Winter 2020
Exam 1 Practice Problems

These are problems like those that might occur on an exam. This collection of problems is not intended to have the same length, or cover precisely the same topics, as the actual exam.

1. TRUE or FALSE? (Note that TRUE means always true, and FALSE means not always true.)

   (a) If the power series $\sum_{k=0}^{\infty} c_k x^k$ converges for $x = 2$, then the power series $\sum_{k=0}^{\infty} k c_k x^{k-1}$ converges for $x = -1$.

   (b) If $F(t)$ is the force acting on an object at time t, then the work done on the object between times $t = a$ and $t = b$ can be computed as $\int_a^b F(t)\,dt$.

   (c) Suppose $(a_n)_{n=1}^{\infty}$ is a sequence and there is $N \in \mathbb{N}$ such that if $n > N$ then
   $$|a_n - 4| < 0.00005.$$
   Then (a_n) converges to 4.

   (d) When finding the volume of the region determined by revolving the region A around an axis, you must use the volumes by washers method if you are revolving around the x-axis, and the volumes by shells method if you are revolving around the y-axis.

   (e) Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$. Then $\sum_{n=0}^{\infty} a_n$ diverges.

   (f) A function has a power series representation $f(x) = \sum_{n=0}^{\infty} c_n (x-2)^n$ with a radius of convergence $R = 1$. We can use the power series to estimate $f(3)$ correctly to within any error.

   (g) If $\int_0^1 f(x)\,dx = 2$ and $\int_0^1 g(x)\,dx = 3$, then $\int_0^1 f(x)g(x)\,dx = 6$.

   (h) If $P(x)$ is the probability density function for a random method of choosing a real number, then $P(x)$ is the probability of choosing x.

2. Multiple choice and short answer problems:

(a) Suppose $a_0 = 1$ and $0 \le a_n \le \dfrac{1}{2^n}$ for every n, and you know that $\displaystyle\sum_{n=0}^{\infty} a_n$ converges to one of the following numbers. Which is correct? Circle the correct answer. (Note that $\pi \approx 3.14$.)

i. $\left(\dfrac{\pi}{4}\right)$

ii. $\left(\dfrac{\pi}{2}\right)$

iii. (π)

iv. (2π)

(b) If $0 \le a_k \le \dfrac{4}{2^k}$ for all k, and the series $\displaystyle\sum_{k=0}^{\infty} a_k$ converges to π, find a bound on the error in using $\displaystyle\sum_{k=0}^{5} a_k$ as an approximation for π.

(c) The 2-dimensional region bounded by the curves $y = x^3$, $x = 0$, and $y = 1$ is revolved around the x-axis, to sweep out a 3-dimensional region.

i. Sketch the 2-dimensional region being revolved.

ii. Write down an integral that represents the volume of the 3-dimensional region, computed using volumes by discs or washers. You do NOT have to evaluate this integral.

iii. Write down an integral that represents the volume of the 3-dimensional region, computed using volumes by shells. You do NOT have to evaluate this integral.

(d) The same as 2(c), except revolve the region around the y-axis.

3. Suppose you know that $\ln 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \dfrac{1}{5} - \dfrac{1}{6} + \cdots$. (This is true.)

(a) Find a bound on the error in the approximation: $\ln 2 \approx 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} = \dfrac{7}{12}$.

(b) Suppose you want to approximate $\ln 2$ correct to 1 decimal place (error at most .05), as a finite partial sum of $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots$. What is the last term you will include? (Give the term itself. For example, for the approximation in part (a), you should say the last term included was $-\dfrac{1}{4}$, not "the fourth term.")

4. A metal pyramid whose base is a 10 centimeter by 10 centimeter square, and whose height is 10 centimeters, is made of metal that fades from nearly pure gold at the top to nearly pure silver at the bottom. The price of the metal x centimeters below the top point of the pyramid is $(915 - 90x)$ dollars per cubic centimeter.

Approximate the total cost of the metal in the pyramid using a Riemann sum. Be sure to explain your reasoning. (Hint: Divide the pyramid into horizontal slices.)

Express the total cost of the metal in the pyramid as a definite integral. You do NOT have to evaluate the integral. Be sure to include units in your answer.

5. A function $f(x)$ has the following derivatives at $x = 1$:

$$f^{(k)}(1) = \begin{cases} 0 & \text{if } k \text{ is odd;} \\ k & \text{if } k \text{ is even.} \end{cases}$$

   (a) Write down the degree 8 Taylor polynomial for f centered at 1.

   (b) Write down the Taylor series for f centered at 1 using summation $\left(\sum\right)$ notation.

   (c) Find the radius of convergence of this series.

6. The horizontal base of a three-dimensional object is the region given by $1 \leq x \leq e$ and $0 \leq y \leq \ln x$. The cross-section of the object perpendicular to the x-axis at point x is a rectangle of height x. Find the volume of the object.

7. What is the average value of e^{-x} between 0 and 2π?

8. Use the definition of limit to show that $\displaystyle\lim_{n\to\infty} \frac{4n}{2n+1} = 2$.