Math 8
Winter 2020
Exam 1 Practice Problems

These are problems like those that might occur on an exam. This collection of problems is not intended to have the same length, or cover precisely the same topics, as the actual exam.

1. TRUE or FALSE? (Note that TRUE means always true, and FALSE means not always true.)

   (a) If the power series $\sum_{k=0}^{\infty} c_k x^k$ converges for $x = 2$, then the power series $\sum_{k=0}^{\infty} k c_k x^{k-1}$ converges for $x = -1$.

   TRUE: Since the first power series is centered at 0 and converges for $x = 2$, its radius of convergence R must be greater than or equal to 2. The second power series is the derivative of the first, so it has the same center 0 and radius of convergence $R \geq 2$, and it must converge for all x between -2 and 2.

   (b) If $F(t)$ is the force acting on an object at time t, then the work done on the object between times $t = a$ and $t = b$ can be computed as $\int_a^b F(t)\, dt$.

   FALSE: Work is the integral of force with respect to distance, not with respect to time.

   (c) Suppose $(a_n)_{n=1}^{\infty}$ is a sequence and there is $N \in \mathbb{N}$ such that if $n > N$ then

   $$|a_n - 4| < 0.00005.$$

   Then (a_n) converges to 4.

   FALSE: For example, the constant sequence $a_n = 4.000000001$ satisfies this condition ($|a_n - 4| < 0.00005$ for every n) but it does not converge to 4

   (d) When finding the volume of the region determined by revolving the region A around an axis, you must use the volumes by washers method if you are revolving around the x-axis, and the volumes by shells method if you are revolving around the y-axis.

   FALSE: Either technique can be used for either sort of problem. (If one produces an integral you cannot compute, try the other.)

   (e) Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$. Then $\sum_{n=0}^{\infty} a_n$ diverges.

   FALSE: For example, the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ satisfies $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, but it converges by the alternating series test.

(f) A function has a power series representation $f(x) = \sum_{n=0}^{\infty} c_n(x-2)^n$ with a radius of convergence $R = 1$. We can use the power series to estimate $f(3)$ correctly to within any error.

FALSE: At a distance R from the center, the series may or may not converge. For example, the function $f(x) = \frac{1}{x-1}$ has a power series representation $\sum_{n=0}^{\infty}(-1)^n(x-2)^n$ with radius of convergence 1, but if we substitute $x = 3$ to try to estimate $f(3)$, we get the series $\sum_{n=0}^{\infty}(-1)^n$, whose partial sums alternate between 1 and 0, and do not approach $f(3)$, which is $\frac{1}{2}$.

(g) If $\int_0^1 f(x)\,dx = 2$ and $\int_0^1 g(x)\,dx = 3$, then $\int_0^1 f(x)g(x)\,dx = 6$.

FALSE: The integral of the product is not the product of the integrals. (This is why we need integration by parts.)

For example, $\int_0^1 4x\,dx = 2$ and $\int_0^1 (6-6x)\,dx = 3$, but $\int_0^1 4x(6-6x)\,dx = 4$.

(h) If $P(x)$ is the probability density function for a random method of choosing a real number, then $P(x)$ is the probability of choosing x.

FALSE: $P(x)$ is the probability density function at x. This means that for a small interval of length Δx containing x, the probability of choosing a number in that interval is approximately $P(x)\,\Delta x$.

2. Multiple choice and short answer problems:

(a) Suppose $a_0 = 1$ and $0 \le a_n \le \frac{1}{2^n}$ for every n, and you know that $\sum_{n=0}^{\infty} a_n$ converges to one of the following numbers. Which is correct? Circle the correct answer. (Note that $\pi \approx 3.14$.)

i. $\left(\frac{\pi}{4}\right)$

ii. $\left(\frac{\pi}{2}\right)$

iii. (π)

iv. (2π)

Solution: (ii) $\frac{\pi}{2}$. Because $a_0 = 1$ and all a_n are positive, we know $\sum_{n=0}^{\infty} a_n \ge 1$, and $\frac{\pi}{4} < 1$. Because $a_n \le \frac{1}{2n}$ for every n, we know (by the comparison test)

$\displaystyle\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$ (we can find the sum of this second series because it is geometric), and $\pi > 2$. Therefore, $\frac{\pi}{2}$ is the only possible answer.

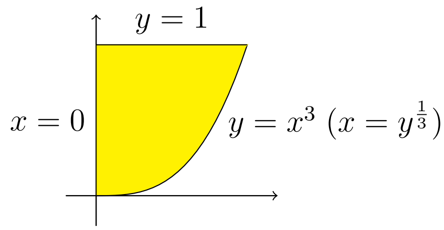
(b) If $0 \leq a_k \leq \frac{4}{2^k}$ for all k, and the series $\displaystyle\sum_{k=0}^{\infty} a_k$ converges to π, find a bound on the error in using $\displaystyle\sum_{k=0}^{5} a_k$ as an approximation for π.

Solution: The error is $\displaystyle\sum_{k=6}^{\infty} a_k$, the sum of the terms not included in the approximation. By the comparison test, this is at most $\displaystyle\sum_{k=6}^{\infty} \frac{4}{2^k}$. This is a geometric series with first term $\frac{4}{2^6} = \frac{1}{16}$ and ratio $\frac{1}{2}$. Therefore its sum is $\frac{\frac{1}{16}}{1-\frac{1}{2}} = \frac{1}{8}$, and this is a bound on the error.

(c) The 2-dimensional region bounded by the curves $y = x^3$, $x = 0$, and $y = 1$ is revolved around the x-axis, to sweep out a 3-dimensional region.

i. Sketch the 2-dimensional region being revolved.

Solution:



ii. Write down an integral that represents the volume of the 3-dimensional region, computed using volumes by discs or washers. You do NOT have to evaluate this integral.

Solution: $\displaystyle\int_0^1 \pi(1-(x^3)^2)\,dx$

iii. Write down an integral that represents the volume of the 3-dimensional region, computed using volumes by shells. You do NOT have to evaluate this integral.

Solution: $\displaystyle\int_0^1 2\pi y(y^{\frac{1}{3}})\,dy$

(d) The same as 2(c), except revolve the region around the y-axis.

Solution:

Discs: $\displaystyle\int_0^1 \pi(y^{\frac{1}{3}})^2\,dy$

Shells: $\displaystyle\int_0^1 2\pi x(1-x^3)\,dx$

3. Suppose you know that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$. (This is true.)

   (a) Find a bound on the error in the approximation: $\ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$.

   Solution: By the alternating series error bound, the error is at most the absolute value of the first term not included, which is $\frac{1}{5}$.

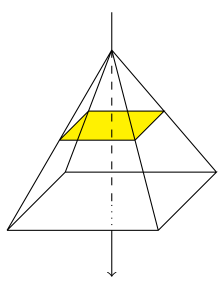
   (b) Suppose you want to approximate $\ln 2$ correct to 1 decimal place (error at most .05), as a finite partial sum of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. What is the last term you will include? (Give the term itself. For example, for the approximation in part (a), you should say the last term included was $-\frac{1}{4}$, not "the fourth term.")

   Solution: By the alternating series error bound again, we want the first term not included to have absolute value at most $.05 = \frac{1}{20}$. Therefore we want the first term not included to be $-\frac{1}{20}$, and the last term included to be $\frac{1}{19}$.

4. A metal pyramid whose base is a 10 centimeter by 10 centimeter square, and whose height is 10 centimeters, is made of metal that fades from nearly pure gold at the top to nearly pure silver at the bottom. The price of the metal x centimeters below the top point of the pyramid is $(915 - 90x)$ dollars per cubic centimeter.

   Approximate the total cost of the metal in the pyramid using a Riemann sum. Be sure to explain your reasoning. (Hint: Divide the pyramid into horizontal slices.)

   Express the total cost of the metal in the pyramid as a definite integral. You do NOT have to evaluate the integral. Be sure to include units in your answer.



   Solution: Draw the x-axis through the center of the pyramid pointing downward, with $x = 0\,\text{cm}$ at the top of the pyramid and $x = 10\,\text{cm}$ at the base. A cross-section $x\,\text{cm}$ below the top, as in the picture, is the base of a similar pyramid with height $x\,\text{cm}$, so it is a square with side measuring $x\,\text{cm}$, and with area $x^2\,\text{cm}^2$.

   Divide the interval $0 \le x \le 10$ into n pieces of length Δx, and for each i choose x_i^* in the i^{th} piece. This divides the pyramid into n horizontal slices of height $\Delta x\,\text{cm}$.

   The i^{th} slice is at height approximately x_i^*, so its volume is approximately the area of the cross-section at x_i^* times $\Delta x\,\text{cm}$, or $(x_i^*)^2\,\Delta x\,\text{cm}^3$.

Metal x_i^* cm below the top of the pyramid costs $(915 - 90x_i^*)$ dollars per cubic centimeter, so the cost of the metal in the i^{th} slice is approximately $(915-90x_i^*)(x_i^*)^2\,\Delta x$ dollars (cost per cubic centimeter times volume).

The total cost of the metal in the pyramid is the sum of the costs of the slices, or approximately

$$\boxed{\sum_{i=1}^{n}(915-90x_i^*)(x_i^*)^2\,\Delta x \text{ dollars.}}$$

To find the cost, we take the limit

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}(915-90x_i^*)(x_i^*)^2\,\Delta x \text{ dollars}\right) = \boxed{\int_0^{10}(915-90x)(x)^2\,dx \text{ dollars}}.$$

5. A function $f(x)$ has the following derivatives at $x = 1$:

$$f^{(k)}(1) = \begin{cases} 0 & \text{if } k \text{ is odd;} \\ k & \text{if } k \text{ is even.} \end{cases}$$

(a) Write down the degree 8 Taylor polynomial for f centered at 1.

Solution:

$$\frac{2(x-1)^2}{2!} + \frac{4(x-1)^4}{4!} + \frac{6(x-1)^6}{6!} + \frac{8(x-1)^8}{8!}.$$

(b) Write down the Taylor series for f centered at 1 using summation $\left(\sum\right)$ notation.

Solution:

$$\sum_{n=0}^{\infty}\frac{2n(x-1)^{2n}}{(2n)!}$$

(c) Find the radius of convergence of this series.

Solution: Use the ratio test.

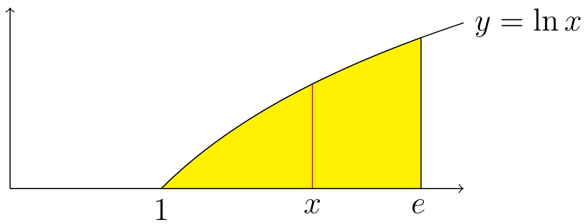
$$\lim_{n\to\infty}\left|\frac{\frac{2(n+1)(x-1)^{2(n+1)}}{(2(n+1))!}}{\frac{2n(x-1)^{2n}}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)}{2n}\frac{(2n)!}{(2n+2)!}(x-1)^2\right| =$$

$$\lim_{n\to\infty}\left|\frac{2n+2}{2n}\frac{1}{(2n+2)(2n+1)}(x-1)^2\right| = (x-1)^2\lim_{n\to\infty}\left|\frac{1}{2n(2n+1)}\right| = 0$$

Since this limit is always less than 1, the series converges everywhere, and the radius of convergence is $R = \infty$.

6. The horizontal base of a three-dimensional object is the region given by $1 \le x \le e$ and $0 \le y \le \ln x$. The cross-section of the object perpendicular to the x-axis at point x is a rectangle of height x. Find the volume of the object.

Solution:



The picture shows the base of the solid. The red line is the base of the cross-section at x. This cross-section is a rectangle with base of this length ($\ln x$) and height x, so the area of this cross-section is $A(x) = x \ln x$. To find the volume, we integrate the cross-sectional area, to get $\int_1^e x \ln x \, dx$.

To find $\int x \ln x \, dx$, use integration by parts with $u = \ln x$ and $dv = x\,dx$, so $du = \frac{1}{x}\,dx$ and $v = \frac{x^2}{2}$. This gives

$$\int x \ln x \, dx = \int u \, dv = uv - \int v \, du = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

$$\text{V}olume = \int_1^e x \ln x \, dx = \left(\frac{x^2 \ln x}{2} - \frac{x^2}{4}\right)\Big|_{x=1}^{x=e} = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}.$$

7. What is the average value of e^{-x} between 0 and 2π?

Solution: $\frac{1}{2\pi - 0}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{2\pi}(-e^{-x})\Big|_{x=0}^{x=2\pi} = \frac{1-e^{-2\pi}}{2\pi}$

8. Use the definition of limit to show that $\lim_{n\to\infty} \frac{4n}{2n+1} = 2$.

Solution: Given $\varepsilon > 0$, we must find N such that whenever $n > N$ we have

$$\left|2 - \frac{4n}{2n+1}\right| < \varepsilon \tag{1}$$

Do some algebra:

$$2 - \frac{4n}{2n+1} = \frac{4n+2}{2n+1} - \frac{4n}{2n+1} = \frac{2}{2n+1}. \tag{2}$$

Do some more algebra (assuming $n \ge 0$):

$$\left|\frac{2}{2n+1}\right| < \varepsilon \iff \frac{2}{2n+1} < \varepsilon \iff \frac{2}{\varepsilon} < 2n+1 \iff \left(\frac{1}{\varepsilon} - \frac{1}{2}\right) < n. \tag{3}$$

Choose any $N \geq \left(\frac{1}{\varepsilon} - \frac{1}{2}\right)$. Then whenever $n > N$, we have $n > \left(\frac{1}{\varepsilon} - \frac{1}{2}\right)$, so by line (3) we have $\left|\frac{2}{2n+1}\right| < \varepsilon$, so by line (2) we have $\left|2 - \frac{4n}{2n+1}\right| < \varepsilon$, which is what we needed (line (1)).

Note: It is fine to simply choose $N = \left(\frac{1}{\varepsilon} - \frac{1}{2}\right)$. Some ways of writing the definition of limit require N to be a whole number, which is why we say choose $N \geq \left(\frac{1}{\varepsilon} - \frac{1}{2}\right)$. However, for this class, either will do.