Practice exam for unit 2

These are problems like those that might occur on an exam. This collection of problems is not intended to have the same length, or cover precisely the same topics, as the actual exam.

1 True or false questions

Indicate whether the following statements are TRUE (T) or FALSE/NOT NECESSARILY TRUE (X). You do not need to justify your answer.

(a) The line with vector equation $\mathbf{r}(t) = \langle 1-2t, -1+6t, 2+4t\rangle$ is orthogonal to the plane $x - 3y - 2z = 1$.

(b) A projectile is fired from a cannon in an arc with velocity given by a vector function $\mathbf{v}(t)$, as a function of time. The distance along the ground that the projectile travels as a function of time can be obtained by integrating its speed.

(c) The vector functions $\mathbf{r}_1(t) = \langle t, te^{2t}\rangle$ and $\mathbf{r}_2(s) = \langle s-1, se^{2s-2} - e^{2s-2}\rangle$ describe the same curve in $\mathbb{R}^2$.

(d) The magnitude of the projection of $\vec{v}$ onto $\vec{w}$ always equals the component of $\vec{v}$ in the direction of $\vec{w}$.

(e) Suppose a particle moves in 3-dimensional space with position vector $\mathbf{r}(t)$. If the particle's velocity $\mathbf{v}(t)$ is parallel to its acceleration $\mathbf{a}(t)$, then it moves in a straight line.

(f) $x^2 + y^2 + 4z^2 = 0$ is the equation of a sphere.

(g) Given a curve with position vector $\mathbf{r}(t)$ parameterized over t, if the derivative of its unit tangent vector is zero everywhere, then the curve is a straight line.

(h) A particle moves in space with position vector

$$\mathbf{r}(t) = \langle \cos(2t), \sin(2t), \cos(2t)\rangle,$$

parameterized by time $t \geq 0$. At any point in time the particle's acceleration is in the opposite direction to its position vector.

Solutions: (a) T, (b) X, (c) T, (d) X, (e) T, (f) X, (g) T, (h) T

2 Multiple choice questions

Choose the best answer.

1. If two lines in $\mathbb{R}^3$ are not parallel, then

   (a) They intersect at a single point.

   (b) They do not intersect.

   (c) Either of the above could be true.

2. The following line is perpendicular to the plane $x - 4y + 3z = 4$.

   (a) $\vec{r} = \langle 1, -4, 3\rangle + t\langle 1, 1, -1\rangle$.

   (b) $\vec{r} = \langle -2, 3, 0\rangle + t\langle 1, 1, 1\rangle$.

   (c) Neither of the above.

3. If ℓ is a line in $\mathbb{R}^3$ in the direction of the vector $\vec{v}$, and $\mathcal{P}$ is a plane in $\mathbb{R}^3$ with normal vector $\vec{n}$, and ℓ intersects $\mathcal{P}$ at a single point, then

   (a) $\vec{v}$ and $\vec{n}$ must be parallel.

   (b) $\vec{v}$ and $\vec{n}$ must be perpendicular.

   (c) $\vec{v}$ and $\vec{n}$ may or may not be parallel, but cannot be perpendicular.

   (d) $\vec{v}$ and $\vec{n}$ may or may not be perpendicular, but cannot be parallel.

Solutions: 1.(c), 2(c), 3.(c)

3 Written problems

1. A moving particle has position function $\vec{r}(t) = \langle t\cos t, t\sin t, t\rangle$.

   (a) Find the object's velocity and acceleration functions $\vec{v}(t)$ and $\vec{a}(t)$.
   Solution: $\vec{v}(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 1\rangle$
   $\vec{a}(t) = \langle -2\sin t - t\cos t, 2\cos t - t\sin t, 0\rangle$.

   (b) Find the object's speed at time t.
   Solution:
   $|\vec{v}(t)| = \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 1} = \sqrt{t^2+2}$.

(c) Write down an integral that gives the arc length of the object's path between the points $(0,0,0)$ and $(-\pi,0,\pi)$. Do not evaluate this integral.

Solution: $\displaystyle\int_0^{\pi} \sqrt{t^2+2}\,dt.$

(d) At time $t=0$, find the object's velocity and acceleration, and the unit tangent vector to the object's path $\vec{T}$.

Solution: $\vec{v}(0) = \langle 1,0,1\rangle$, $\vec{a}(0) = \langle 0,2,0\rangle$, $\vec{T}$ is the unit vector in the direction of $\vec{v}(0)$, or $\vec{T} = \left\langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right\rangle$.

(e) Find the tangential and normal components of the object's acceleration when it is at the point $(0,0,0)$.

Solution: Since $\vec{a}(0)$ is normal to $\vec{v}(0)$, the tangential component of acceleration is $a_{\mathbf{T}} = 0$, and the normal component is $a_{\mathbf{N}} = |\vec{a}(0)| = 2$.

(f) Find the curvature of the object's path at the point $(0,0,0)$.

Solution: At $t=0$ the object's speed is $\frac{ds}{dt} = \sqrt{2}$ and the normal component of acceleration is 2. Using the formula $a_{\mathbf{N}} = \left(\frac{ds}{dt}\right)^2 \kappa$, we get $2 = (\sqrt{2})^2\kappa$, or $\kappa = 1$.

2. Find a function $\vec{r}(t)$ parametrizing the curve formed by the intersection of the surfaces with equations $z = \sqrt{x^2+y^2}$ and $z^2 = 2x+2y+2$.

Solution: Substituting $z = \sqrt{x^2+y^2}$ in the second equation yields $x^2+y^2 = 2x+2y+2$ or $x^2-2x+1+y^2-2y+1 = 4$, or $\left(\frac{x-1}{2}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$. From this we can set $\left(\frac{x-1}{2}\right) = \cos t$ and $\left(\frac{y-1}{2}\right) = \sin t$, or $x = 2\cos t+1$, $y = 2\sin t + 1$. Substituting back into either of our equations gives $z = \sqrt{4\cos t + 4\sin t + 6}$. This gives $\vec{r}(t) = \left\langle 2\cos t + 1,\ 2\sin t + 1,\ \sqrt{4\cos t + 4\sin t + 6}\right\rangle$.

3. Find the work done by the force of gravity $\langle 0,0,-mg\rangle$ on an object of mass m sliding down a straight ramp from the point $(0,0,1)$ to the point $(2,3,0)$.

Solution: The work is the dot product of the force and the displacement, in this case $\langle 0,0,-mg\rangle \cdot \langle 2,3,-1\rangle$, or mg. (Since the problem did not specify units, the answer need not specify units. If the units in the problem were newtons (for force) and meters (for displacement), the units in the answer would be joules.)

4. Find a linear equation of the plane containing the points $A = (1,0,-2)$, $B = (1,3,1)$ and $C = (-1,3,-2)$.

Solution: To find the linear equation we need a normal vector and a point on the plane. A normal vector is given by $\vec{n} = \vec{AB} \times \vec{AC} = \langle 0, 3, 3\rangle \times \langle -2, 3, 0\rangle = \langle -9, -6, 6\rangle$ (You can also use $\vec{AB} \times \vec{BC}$ or $\vec{AC} \times \vec{BC}$ as a normal vector). Now, computing $\langle -9, -6, 6\rangle \cdot \langle x - 1, y, z + 2\rangle = 0$ gives the linear equation $3x + 2y - 2z = 7$.

5. Find a function parametrizing:

   (a) the curve formed by the intersection of the sphere $x^2 + y^2 + z^2 = 10$ and the plane $z = 1$.
   Solution: Substituting $z = 1$ into $x^2 + y^2 + z^2 = 10$ yields the equation $x^2 + y^2 = 9$; that is, a circle with radius 3. Putting $x = 3\cos(t)$ and $y = 3\sin(t)$ will satisfy the equation $x^2 + y^2 = 9$. Thus, $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 1\rangle$ is a parametrization for the required curve.

   (b) the curve formed by the intersection of the planes $2x+y-z = 3$ and $x-2y+z = 0$.
   Solution: Substitute $z = -x + 2y$ into $2x + y - z = 3$ to get $3x - y = 3$. Since there are two unknowns in the latter equation, it follows that there are infinitely many solutions. Therefore we introduce a parameter: let $x = t$. Then $y = 3t - 3$ and $z = 5t-6$. Thus, $\vec{r}(t) = \langle t, 3t-3, 5t-6\rangle$ is a parametrization for the required curve.

6. Let $\vec{u}$ and $\vec{v}$ be unit vectors, and $\vec{u} \cdot \vec{v} = \frac{1}{2}$. What is $|\vec{u} \times \vec{v}|$?
   Solution: Since $\vec{u}$ and $\vec{v}$ are unit vectors, it follows that $\vec{u} \cdot \vec{v} = \cos(\theta) = \frac{1}{2}$. Thus, $\theta = \frac{\pi}{3}$. Now, $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

7. Find the angle B of the triangle whose vertices are $A = (7, 0, 6)$, $B = (5, 0, 4)$ and $C = (4, 1, 2)$.
   Solution: Let θ be the angle formed by the vertex B inside the triangle ABC. Then

$$\cos(\theta) = \frac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}||\vec{BC}|} = \frac{\langle 2, 0, 2\rangle \cdot \langle -1, 1, -2\rangle}{|\langle 2, 0, 2\rangle||\langle -1, 1, -2\rangle|} = -\frac{\sqrt{3}}{2}.$$

   Thus, $\theta = \frac{5\pi}{6}$.

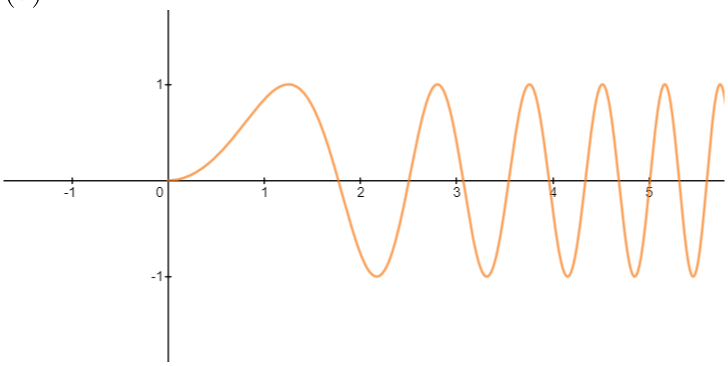
8. At what points does the helix $r(\vec{t}) = \langle \cos(t), \sin(t), t\rangle$ intersect the surface $x^2+y^2+z^2 = 10$?
   Solution: Substituting the vector-valued function's components into the equation of the sphere yields $\cos^2(t) + \sin^2(t) + t^2 = 10$, or $t^2 = 9$, which implies that $t = 3$ or $t = -3$. Evaluating $r(\vec{t})$ for $t = 3$ and $t = -3$ gives the points of intersection $(\cos(3), \sin(3), 3)$ and $(\cos(-3), \sin(-3), -3)$.

9. (a) Sketch the curve $r(\vec{t}) = \langle \sqrt{t}, \sin(t)\rangle$.

   (b) Find a tangent line to the curve at point $(\sqrt{2\pi}, 0)$.

(c) Give a position function of a particle traveling on that curve from $(\sqrt{2\pi}, 0)$ to $(0, 0)$. Don't forget to specify the domain.

Solution:
(a)



(b) The point $(\sqrt{2\pi}, 0)$ corresponds to $t = 2\pi$. We compute the derivative: $\vec{r}'(t) = \langle \frac{1}{2}t^{-1/2}, \cos(t)\rangle$. Thus $\vec{r}'(2\pi) = \langle \frac{1}{2\sqrt{2\pi}}, 1\rangle$ gives the direction of the desired tangent line. Then a vector equation of the tangent line is $\vec{l}(t) = \langle \sqrt{2\pi}, 0\rangle + t\langle \frac{1}{2\sqrt{2\pi}}, 1\rangle$.

(c) $\vec{r_1}(t) = \langle \sqrt{-t}, -\sin(t)\rangle$ for $-2\pi \leq t \leq 0$.

10. An artist needs for a project a long ribbon that will be arranged in the shape of the curve $\langle t^2, \frac{2t^3}{3}, \frac{t^4}{4}\rangle$, with the z-axis pointing up. The total creation should be 4 meters high, and the ribbon will cover the total height of the creation, from the top of the creation to the floor. How long should the ribbon be?

Solution: We need to find the value of t corresponding to a height of 4 meters. This means the z-component needs to be 4; that is, $\frac{t^4}{4} = 4$. Then $t = 2$ or $t = -2$ are both solutions, but since the ribbon will only cover the creation all the way to the floor (i.e. $t = 0$), we are only interested in $0 \leq t \leq 2$. Now the length of the desired ribbon is

$$\begin{aligned}
L &= \int_0^2 \sqrt{(2t)^2 + (2t^2)^2 + (t^3)^2}\, dt \\
&= \int_0^2 t\sqrt{4 + 4t^2 + t^4}\, dt \qquad \left(\text{or } = \int_0^2 \sqrt{(2t + t^3)^2}\, dt\right) \\
&= \int_0^2 t\sqrt{(2 + t^2)^2}\, dt = \int_0^2 t(2 + t^2)\, dt \\
&= \left[t^2 + \frac{1}{3}t^3\right]_0^2 \\
&= \frac{20}{3}.
\end{aligned}$$

11. (a) Give a function parametrizing the curve $\frac{(x-2)^2}{16} + \frac{y^2}{9} = 4$.

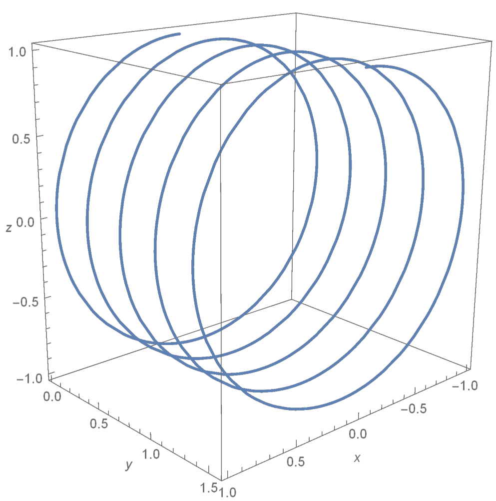
(b) Describe this curve in words. Be as specific as possible.

Solution: (a) Dividing both sides by 4 yields the ellipse $\dfrac{(x-2)^2}{8^2} + \dfrac{y^2}{6^2} = 1$. Letting $\cos(t) = \frac{(x-2)}{8}$ and $\sin(t) = \frac{y}{6}$ and solving for x and y yields $x = 8\cos(t) + 2$ and $y = 6\sin(t)$. Then $\vec{r}(t) = \langle 8\cos(t) + 2, 6\sin(t)\rangle$ parametrizes the given curve.

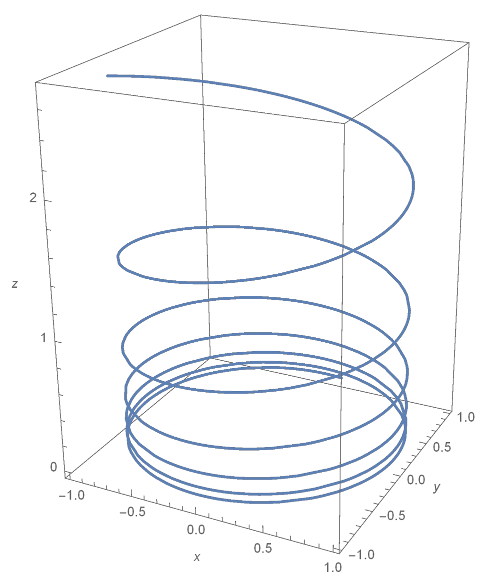
(b) The curve is an ellipse with center $(2,0)$, has x- and y-radii of 8 and 6, respectively, the x-intercepts are $(10,0)$ and $(-6,0)$, and the y-intercepts are $(0, -\frac{3\sqrt{15}}{2})$ and $(0, \frac{3\sqrt{15}}{2})$.

4 Picture matching problem

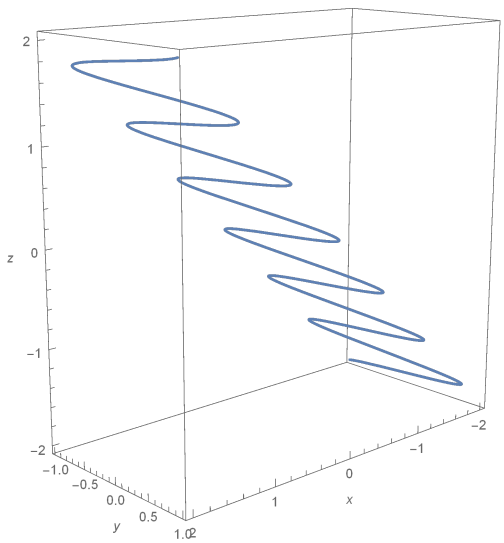
Match the following pictures with their description. Note that there are more options to choose from than pictures.



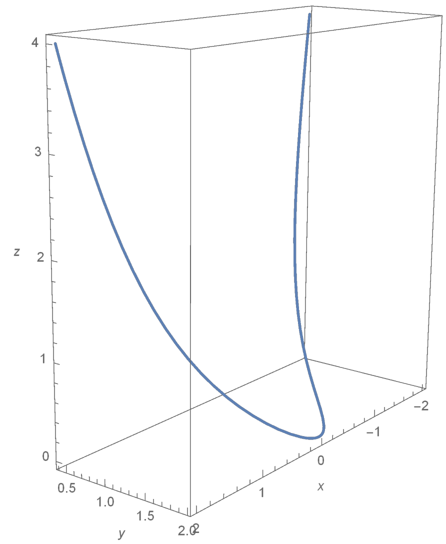
a)



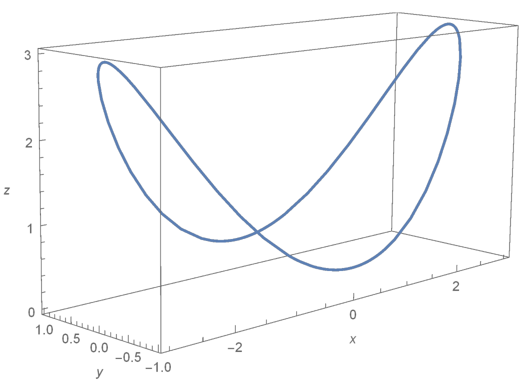
b)



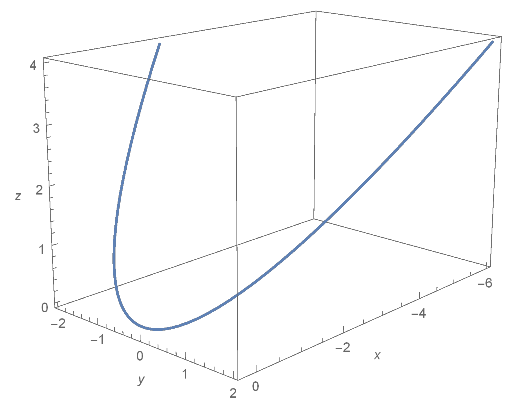
c)



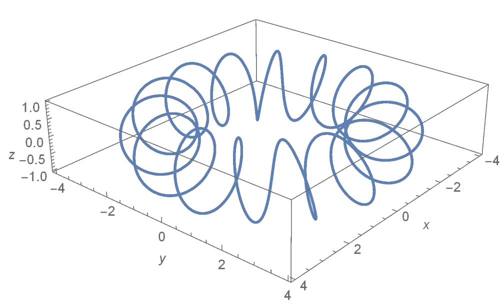
d)



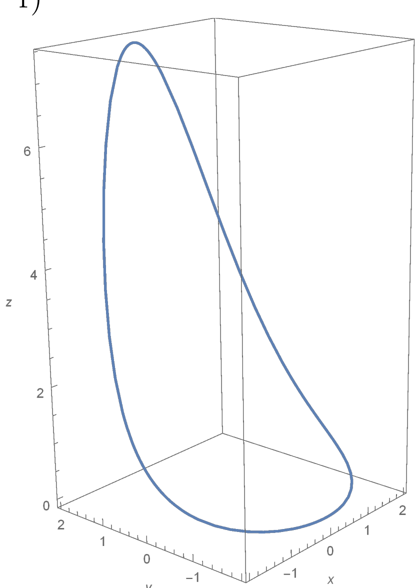
e)



f)



g)



h)

1. $z^2 = 1 + x^2 + y^2$
2. The intersection of $y^2 = z$ and the plane containing the origin and normal to the vector $\vec{v} = \langle 1, 1, 1 \rangle$
3. $\frac{x^2}{4} + y^2 + z = 0$
4. $\frac{x^2}{9} + \frac{(y-2)^2}{4} + (z-3)^2 = 1$
5. The intersection of $x = z$ and $y = \sin(x)$
6. The cylinder whose trace in the x-y plane is $\frac{x^2}{4} + y^2 = 1$
7. $\vec{r}(t) = \langle \cos(t), \sin(t), e^t \rangle$
8. The intersection of $x^2 + y^2 = 2$ and $z = e^y$
9. $\frac{x^2}{4} + y^2 - z^2 = 1$
10. The intersection of the the paraboloid $3z = x^2 + y^2$ and the cylinder $\frac{x^2}{9} + y^2 = 1$
11. The sphere with radius 3 centered at $(0, 2, 3)$
12. $x = \sin(2z)$
13. $\vec{r}(t) = \langle (3 + \sin(15t)) \cos(t), (3 + \sin(15t)) \sin(t), \cos(15t) \rangle$
14. The plane containing the origin with a normal vector $\langle 1, -3, 1 \rangle$
15. $\vec{r}(t) = \langle \sin(t), t, \cos(t) \rangle$
16. $\vec{r}(t) = \langle t, \frac{2}{1+t^2}, t^2 \rangle$

Solutions: (a)-15, (b)-7, (c)-5, (d)-16, (e)-10, (f)-2, (g)-13, (h)-8