Math 8
Winter 2020
Final Exam Practice Problems

These are problems like those that might occur on an exam. This collection of problems is not intended to have the same length, or cover precisely the same topics, as the actual exam. In particular, these problems concentrate on the last unit of the course (as will the final exam), and do not at all represent the range of possible problems from the first two units.

1. (Short answer.) TRUE or FALSE? (Mark the statement TRUE if it is always true, and FALSE if it is not always true.)

   (a) If the value of $f(x,y)$ approaches 35 as (x,y) approaches $(0,0)$ along any straight line through the origin, then $\lim_{(x,y)\to(0,0)} f(x,y) = 35$.

   (b) If the function $f(x,y)$ is differentiable everywhere and $\nabla f(x,y)$ is never zero, then the maximum and minimum values of $f(x,y)$ on the disk $x^2+y^2 \le 1$ must occur on the circle $x^2+y^2=1$.

   (c) If $f(x,y)$ is a differentiable function, γ is a curve with a regular (smooth) parametrization and endpoints (x_0,y_0) and (x_1,y_1), and $\nabla f(x,y)$ is zero at every point of γ, then $f(x_0,y_0)=f(x_1,y_1)$.

   (d) The points $(0,1)$ and $(2,1)$ are on the same level curve (contour line) of the function $f(x,y)=x-xy$.

2. (Short answer.) Suppose $\sum_{i=0}^{\infty} c_i(x-a)^i$ is a Taylor series that converges for $x=2$ and diverges for $x=0$. Which of the following can we conclude? Circle **ALL** correct answers.

   (a) The series diverges for $x=-2$.

   (b) The series converges for $x=2.5$.

   (c) The radius of convergence is less than or equal to 2.

   (d) The radius of convergence is not infinity.

   (e) $a \ge 1$.

3. (Short answer.) The functions $f(x,y)=x^2-y^2$ and $\mathcal{P}(x,y)=2x-4y+3$ have the same value -3 when $(x,y)=(1,2)$. To show their graphs are tangent at the point $(1,2,-3)$, we need to show that a certain limit is equal to 0. Write down that limit. (You do *not* need to show it equals 0.)

4. (Short answer.) If you have the following information about the gradient of a function $f(x, y, z)$, what can you say about the level surfaces of f?

   For each function f, choose one option from each group. (The first group refers to the shape of the level surfaces of f, the second group refers to the spacing between level surfaces of f.)

   (a) $\nabla f(x, y, z)$ has the same value at every point.

   The level surfaces of f are:

   i. Spheres.
   ii. Planes.
   iii. Cylinders.

   For equally spaced values of f, the level surfaces of f are:

   i. Equally spaced.
   ii. Not equally spaced.
   iii. There is not enough information to determine the spacing.

   (b) $\nabla f(x, y, z)$ points directly away from the origin at every point.

   The level surfaces of f are:

   i. Spheres.
   ii. Planes.
   iii. Cylinders.

   For equally spaced values of f, the level surfaces of f are:

   i. Equally spaced.
   ii. Not equally spaced.
   iii. There is not enough information to determine the spacing.

   (c) $\nabla f(x, y, z) = \langle x, y, 0 \rangle$ at every point.

   The level surfaces of f are:

   i. Spheres.
   ii. Planes.
   iii. Cylinders.

   For equally spaced values of f, the level surfaces of f are:

   i. Equally spaced.
   ii. Not equally spaced.
   iii. There is not enough information to determine the spacing.

5. Give a function parametrizing the curve that is the intersection of the surfaces described by the equations $3x - z = 2$ and $10x^2 + 10y^2 = z^2$.

6. Using tangent planes, approximate $f(\pi + 0.02, 0.97)$ when
$$f(x, y) = \tan(x) - \ln\left(\frac{x}{y}\right) + y^2.$$

7. A particle moves through space with position function $\vec{r}(t) = \langle e^t, e^t \sin(t), e^t \cos(t)\rangle$ as a function of time $t \geq 0$.

   (a) Compute the distance that the particle travels in the first two seconds.

   (b) Find the curvature of the particle's path at time $t = \pi$.

   (c) Find the normal component of the particle's acceleration at time $t = \pi$.

8. In each case find the limit or show that it does not exist. Justify your answers. You do not need to use the formal (epsilon-delta) definition of limit.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$

   (b) $\displaystyle\lim_{(x,y)\to(0,\pi)} \frac{(x+1)\cos(y)}{e^x}$

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

9. Two hikers Tasha and Chris climb a mountain whose surface can be approximated by the differentiable function
$$f(x, y) = -\frac{1}{4}(x^2 - 2x + 1) - \frac{1}{9}y^2 + 5,$$
where $f(x, y)$ represents the height above sea level. Both climbers start their hike at $(1, -3, 4)$.

   (a) Sketch the level curve of the mountain at the height where they begin their hike. On your picture, draw two arrows indicating possible directions in which the height may change the fastest from the point $(1, -3)$. You should be able to draw these arrows just from the picture of the level curve, without computing anything.

   (b) Chris decides to start walking from $(1, -3, 4)$ in the direction given by vector $2\vec{i}+\vec{j}$ in the xy-plane. Does he ascend or descend, and at what rate?

   (c) From the position $(1, -3, 4)$, in which direction should Tasha walk to ascend fastest? (Give the direction as a unit vector in the xy-plane.)

   (d) What is the rate at which Tasha ascends if she chooses the direction in (c)?

10. A sound source at the origin produces a sound at point (x, y, z) (in meters) whose intensity (in decibels) is

$$f(x, y, z) = \frac{100}{x^2 + y^2 + z^2}.$$

A fly moves with position function $\vec{r}(t) = \langle t, 3 + \cos t, \sin t \rangle$, where t is time in seconds, and the units of position are meters. How fast is the sound intensity experienced by the fly changing when $t = 0$?

11. Consider the function defined by

$$f(x, y) = -2x^2 - 8y^2 + 10.$$

(a) Find the critical points of f.

(b) Using an appropriate test, classify the critical points of f as local maxima, local minima or neither.

(c) Find the absolute maximum and minimum of f inside the region $(x-1)^2+4y^2 \leq 4$.

12. An object moves along the curve parametrized by $\vec{r}(t) = \langle t, t^2 \rangle$, where time t is in seconds and the components of $\vec{r}(t)$ are in meters. The temperature at point (x, y) is given by a differentiable function $f(x, y)$ (in degrees Celsius) with gradient $\nabla f(x, y) = \langle x, 2y \rangle$.

(a) At time t find the object's speed, and a unit vector $\vec{u}$ in the direction of the object's motion.

(b) Use your answer to part (a) to estimate approximately how far the object moves between times t and $t + \Delta t$.

(c) At time t find the directional derivative $D_{\vec{u}} f(x, y)$ of the temperature function, at the location $(x, y) = \vec{r}(t)$, in the direction $\vec{u}$ of the object's motion. Your answer should be a function of t.

(d) Use your answers to earlier parts to estimate the change in the object's temperature between times t and $t + \Delta t$.

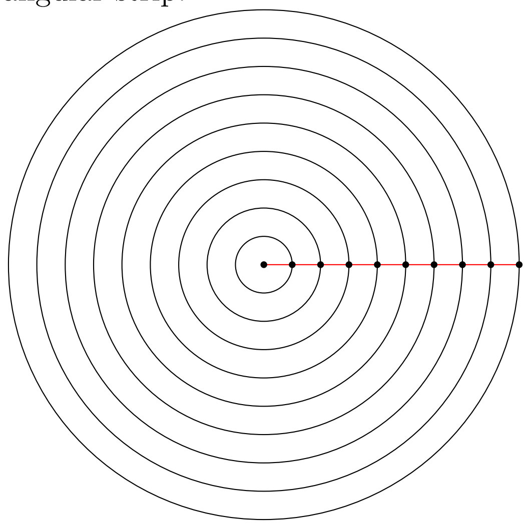
(e) Use your answers to earlier parts to write down a Riemann sum approximating the object's change in temperature between times $t = 0$ and $t = 1$.

(f) Use your answers to earlier parts to write down an integral giving the object's change in temperature between times $t = 0$ and $t = 1$.

Note: Be sure to include units, and explain your answers. Each part of this problem will be graded based on whether it follows correctly from earlier parts.

13. A thin disc of radius π meters is made of material of varying density. At a point r meters from the center of the disc, the mass density per unit area of the disc is $100 + r^2$ grams per square meter. You wish to find the total mass of the disc.

Begin by dividing up the disc into pieces and estimating the mass of each piece. It will be easiest to make your pieces in the form of thin rings, and consider each ring to be approximately a rectangular strip.



(a) Approximately what is the area of the i^{th} ring? Be sure to explain your answer.

(b) Approximately what is the mass of the i^{th} ring? Be sure to explain your answer.

(c) (Short answer.) Write down a Riemann sum approximating the mass of the disc.

(d) (Short answer.) Write down an integral giving the mass of the disc.

Note: Each part of this problem will be graded based on whether it follows correctly from the earlier parts. If your integral in part (d) does give the mass of the disc, but that integral does not follow correctly from your answer to part (c), then you will not get credit for part (d).