Math 8
Winter 2020
Final Exam Practice Problems

These are problems like those that might occur on an exam. This collection of problems is not intended to have the same length, or cover precisely the same topics, as the actual exam. In particular, these problems concentrate on the last unit of the course (as will the final exam), and do not at all represent the range of possible problems from the first two units.

1. (Short answer.) TRUE or FALSE? (Mark the statement TRUE if it is always true, and FALSE if it is not always true.)

   (a) If the value of $f(x,y)$ approaches 35 as $(x,y)$ approaches $(0,0)$ along any straight line through the origin, then $\lim_{(x,y)\to(0,0)} f(x,y) = 35$.

   (b) If the function $f(x,y)$ is differentiable everywhere and $\nabla f(x,y)$ is never zero, then the maximum and minimum values of $f(x,y)$ on the disk $x^2+y^2 \leq 1$ must occur on the circle $x^2+y^2=1$.

   (c) If $f(x,y)$ is a differentiable function, $\gamma$ is a curve with a regular (smooth) parametrization and endpoints $(x_0,y_0)$ and $(x_1,y_1)$, and $\nabla f(x,y)$ is zero at every point of $\gamma$, then $f(x_0,y_0) = f(x_1,y_1)$.

   (d) The points $(0,1)$ and $(2,1)$ are on the same level curve (contour line) of the function $f(x,y) = x - xy$.

   **Solution:** (a) X, (b) T, (c) T, (d) T

2. (Short answer.) Suppose $\sum_{i=0}^{\infty} c_i(x-a)^i$ is a Taylor series that converges for $x=2$ and diverges for $x=0$. Which of the following can we conclude? Circle **ALL** correct answers.

   (a) The series diverges for $x=-2$.

   (b) The series converges for $x=2.5$.

   (c) The radius of convergence is less than or equal to 2.

   (d) The radius of convergence is not infinity.

   (e) $a \geq 1$.

   **Answers:** (a), (d), and (e).

3. (Short answer.) The functions $f(x,y) = x^2 - y^2$ and $\mathcal{P}(x,y) = 2x - 4y + 3$ have the same value $-3$ when $(x,y) = (1,2)$. To show their graphs are tangent at the point $(1,2,-3)$, we need to show that a certain limit is equal to 0. Write down that limit. (You do *not* need to show it equals 0.)

   **Answer:** $\displaystyle \lim_{(x,y)\to(1,2)} \frac{f(x,y) - \mathcal{P}(x,y)}{\sqrt{(x-1)^2 + (y-2)^2}} = \lim_{(x,y)\to(1,2)} \frac{x^2 - y^2 - 2x + 4y - 3}{\sqrt{(x-1)^2 + (y-2)^2}}$

4. (Short answer.) If you have the following information about the gradient of a function $f(x,y,z)$, what can you say about the level surfaces of $f$?

   For each function $f$, choose one option from each group. (The first group refers to the shape of the level surfaces of $f$, the second group refers to the spacing between level surfaces of $f$.)

   (a) $\nabla f(x,y,z)$ has the same value at every point.
   The level surfaces of $f$ are:

   i. Spheres.
   ii. Planes.
   iii. Cylinders.

   For equally spaced values of $f$, the level surfaces of $f$ are:

   i. Equally spaced.
   ii. Not equally spaced.
   iii. There is not enough information to determine the spacing.

   **Answer:** ii., then i.

   (b) $\nabla f(x,y,z)$ points directly away from the origin at every point.
   The level surfaces of $f$ are:

   i. Spheres.
   ii. Planes.
   iii. Cylinders.

   For equally spaced values of $f$, the level surfaces of $f$ are:

   i. Equally spaced.
   ii. Not equally spaced.
   iii. There is not enough information to determine the spacing.

   **Answer:** i., then iii. (The answer to the second question would be ii. if the gradient was continuous.)

   (c) $\nabla f(x,y,z) = \langle x, y, 0\rangle$ at every point.
   The level surfaces of $f$ are:

i. Spheres.
ii. Planes.
iii. Cylinders.

For equally spaced values of $f$, the level surfaces of $f$ are:

i. Equally spaced.
ii. Not equally spaced.
iii. There is not enough information to determine the spacing.

**Answer:** iii., then ii.

5. Give a function parametrizing the curve that is the intersection of the surfaces described by the equations $3x - z = 2$ and $10x^2 + 10y^2 = z^2$.

   **Solution:** From the first equation, we have $z = 3x-2$, which can be put into the second one, to get $(x+6)^2 + 10y^2 = 40$. That is the equation of an ellipse centered at $(-6, 0)$ with $x$-radius $\sqrt{40}$ and $y$-radius 2. Along with the fact that $z = 3x + 2$, the equation of the curve describing the intersection is $\langle\sqrt{40}\cos(t) - 6, 2\sin(t), 3\sqrt{40}\cos(t) - 20\rangle$.

6. Using tangent planes, approximate $f(\pi + 0.02, 0.97)$ when

   $f(x, y) = \tan(x) - \ln\left(\frac{x}{y}\right) + y^2.$

   **Solution:** Using the tangent plane approximation, we can compute the value of $f(x, y)$ for $(x, y)$ close to $(\pi, 1)$: this is $f(x, y) \approx f(\pi, 1) + f_x(\pi, 1)(x - \pi) + f_y(\pi, 1)(y - 1) = (-\ln(\pi) + 1) + (\sec^2(\pi) - \frac{1}{\pi})(x - \pi) + 3(y - 1)$. At $(\pi + 0.02, 0.97)$, this is $-\ln(\pi) + (1 - \frac{1}{\pi})0.02 + 0.91$.

7. A particle moves through space with position function $\vec{r}(t) = \langle e^t, e^t\sin(t), e^t\cos(t)\rangle$ as a function of time $t \geq 0$.

   (a) Compute the distance that the particle travels in the first two seconds.
   (b) Find the curvature of the particle's path at time $t = \pi$.
   (c) Find the normal component of the particle's acceleration at time $t = \pi$.

   **Solution:** (a) We compute: $\vec{r}'(t) = \langle e^t, e^t(\sin(t) + \cos(t)), e^t(\cos(t) - \sin(t))\rangle$, and thus

   $$|\vec{r}'(t)| = \sqrt{e^{2t} + e^{2t}(\cos(t) + \sin(t))^2 + e^{2t}(\cos(t) - \sin(t))^2} = \sqrt{3}e^t.$$

   Now the distance is
   $$d = \int_0^2 \sqrt{3}e^t\,dt = \sqrt{3}(e^2 - 1).$$

(b) The unit tangent vector is $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{\sqrt{3}}\langle 1, \sin(t)+\cos(t), \cos(t)-\sin(t)\rangle$, which in turn yields $\vec{T}'(t) = \frac{1}{\sqrt{3}}\langle 0, \cos(t)-\sin(t), -\sin(t)-\cos(t)\rangle$. Now the curvature at $t=\pi$ is

$$\kappa(\pi) = \frac{|\vec{T}'(\pi)|}{|\vec{r}'(\pi)|} = \frac{\sqrt{2}}{3}\,e^{-\pi}.$$

(c) The normal component is:

$$a_{\mathbf{N}} = |\vec{r}'(\pi)|^2\kappa(\pi) = \sqrt{2}\,e^{\pi}$$

Alternate solution to (b) and (c): The acceleration is $\vec{a}(t) = \langle e^t, 2e^t\cos t, -2e^t\sin t\rangle$. At $t=\pi$ the velocity and acceleration are $\vec{v} = \langle e^\pi, -e^\pi, -e^\pi\rangle$ and $\vec{a} = \langle e^\pi, -2e^\pi, 0\rangle$. The tangential part of the acceleration is the projection of $\vec{a}$ in the direction of $\vec{v}$, which is

$$proj_{\vec{v}}(\vec{a}) = \frac{\vec{a}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} = \frac{3e^{2\pi}}{3e^{2\pi}}\langle e^\pi, -e^\pi, -e^\pi\rangle = \langle e^\pi, -e^\pi, -e^\pi\rangle.$$

The normal part of the acceleration is $\vec{a}$ minus the tangential part, or

$$\vec{a} - proj_{\vec{v}}(\vec{a}) = \langle e^\pi, -2e^\pi, 0\rangle - \langle e^\pi, -e^\pi, -e^\pi\rangle = \langle 0, -e^\pi, e^\pi\rangle,$$

and the normal component of acceleration is the magnitude of this part, or $\sqrt{2}\,e^\pi$.

The normal component of acceleration is also the speed squared times the curvature, $a_{\mathbf{N}} = |\vec{r}'(t)|^2\kappa$. At our point we get $\sqrt{2}\,e^\pi = \left(\sqrt{3\,e^{2\pi}}\right)^2\kappa$, and we can solve to get $\kappa = \frac{\sqrt{2}}{3}\,e^{-\pi}$.

8. In each case find the limit or show that it does not exist. Justify your answers. You do not need to use the formal (epsilon-delta) definition of limit.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$

(b) $\displaystyle\lim_{(x,y)\to(0,\pi)} \frac{(x+1)\cos(y)}{e^x}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

**Solution:** (a) Along the $x$-axis or $y$-axis the limit is 1, but along the line $y=x$ the limit is 2. Therefore the limit does not exist.

(b) The function is a quotient of continuous functions such that the denominator is non-zero at $(0,\pi)$. Therefore we may evaluate the limit by directly substituting in the point $(0,\pi)$:

$$\lim_{(x,y)\to(0,\pi)} \frac{(x+1)\cos(y)}{e^x} = \frac{(0+1)\cos(\pi)}{e^0} = -1.$$

(c) Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then

$$\frac{\sin(x^2+y^2)}{x^2+y^2} = \frac{\sin(r^2)}{r^2},$$

which is independent of $\theta$. From limits of single variable functions we know hat $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, and therefore

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r^2)}{r^2} = 1.$$

For (c) you can also substitute directly $u = x^2 + y^2$, note that as $(x, y) \to (0, 0)$ we have $u \to 0$, and evaluate $\lim_{u\to 0} \frac{\sin u}{u}$, which you can do (if you don't remember this limit) using l'Hôpital's rule.

9. Two hikers Tasha and Chris climb a mountain whose surface can be approximated by the differentiable function

$$f(x,y) = -\frac{1}{4}(x^2 - 2x + 1) - \frac{1}{9}y^2 + 5,$$

where $f(x, y)$ represents the height above sea level. Both climbers start their hike at $(1, -3, 4)$.

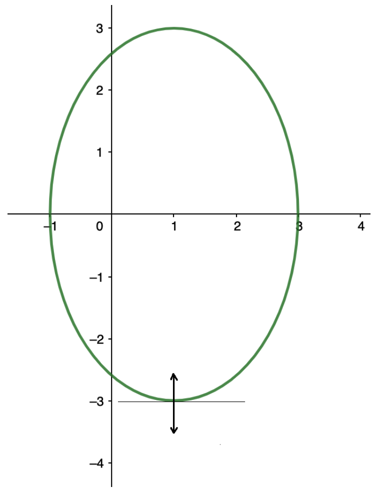
(a) Sketch the level curve of the mountain at the height where they begin their hike. On your picture, draw two arrows indicating possible directions in which the height may change the fastest from the point $(1, -3)$. You should be able to draw these arrows just from the picture of the level curve, without computing anything.

(b) Chris decides to start walking from $(1, -3, 4)$ in the direction given by vector $2\vec{i} + \vec{j}$ in the $xy$-plane. Does he ascend or descend, and at what rate?

(c) From the position $(1, -3, 4)$, in which direction should Tasha walk to ascend fastest? (Give the direction as a unit vector in the $xy$-plane.)

(d) What is the rate at which Tasha ascends if she chooses the direction in (c)?

**Solution:** (a) $f(x, y) = 4$ yields the ellipse $\frac{1}{4}(x-1)^2 + \frac{1}{9}y^2 = 1$. The direction of fastest rate of change will be perpendicular to the tangent at $(1, -3)$.




(b) We need to compute the directional derivative in the direction of $2\vec{i}+\vec{j}$. For this we need a unit vector in the direction of $2\vec{i}+\vec{j}$ and $\vec{u}=\frac{1}{\sqrt{5}}\langle 2,1\rangle$ will do the job. Now we compute the directional derivative at $(1,-3)$:

$$D_{\vec{u}}f(1,-3)=f_x(1,-3)\frac{2}{\sqrt{5}}+f_y(1,-3)\frac{1}{\sqrt{5}}=\frac{2}{3}\sqrt{5}.$$

(c) She should walk in the direction given by $\nabla f(1,-3)=\langle 0,\frac{2}{3}\rangle$, that is, in the direction of the unit vector $\langle 0,1\rangle$.

(d) At a rate of $|\nabla f(1,-3)|=\frac{2}{3}$.

10. A sound source at the origin produces a sound at point $(x,y,z)$ (in meters) whose intensity (in decibels) is

$$f(x,y,z)=\frac{100}{x^2+y^2+z^2}.$$

A fly moves with position function $\vec{r}(t)=\langle t,3+\cos t,\sin t\rangle$, where $t$ is time in seconds, and the units of position are meters. How fast is the sound intensity experienced by the fly changing when $t=0$?

**Solution:** We are looking for the directional derivative of $f$ in the direction of the motion of the fly. At time $t=0$, the direction of motion is $\langle 1,0,1\rangle$, and the position is $\langle 0,4,0\rangle$. The gradient of $f$ at this position is $\langle 0,\frac{25}{8},0\rangle$. Since the gradient and the direction or the motion are orthogonal, the sound is not changing.

11. Consider the function defined by

$$f(x,y)=-2x^2-8y^2+10.$$

(a) Find the critical points of $f$.

(b) Using an appropriate test, classify the critical points of $f$ as local maxima, local minima or neither.

(c) Find the absolute maximum and minimum of $f$ inside the region $(x-1)^2+4y^2 \leq 4$.

**Solution:** (a) The only critical point of $f$ is when both partial derivatives are 0, that is at $(0,0)$.
(b) Here, we can use the second derivative test. The discriminant is 64 at $(0,0)$, and the second-order derivative $f_x x$ is negative. Hence, $(0,0)$ is a maximum. Also, the graph of $f$ is an inverted paraboloid, so we know from it that the only critical point is a maximum.
(c) We need to find the extrema on the boundary of the region, on top of the value at the critical point. Using the method of Lagrange multipliers means solving the system of equations

$$\langle -4x, -16y \rangle = \lambda \langle 2(x-1), 8y \rangle, \tag{1}$$
$$(x-1)^2 + 4y^2 = 4. \tag{2}$$

The only two solutions are at $(-1,0)$ and $(3,0)$. We then get the following:

| $(x,y)$ | $f(x,y)$ | min/max/other |
|---|---|---|
| $(0,0)$ | 10 | max |
| $(-1,0)$ | 8 | other |
| $(3,0)$ | $-8$ | min |

12. An object moves along the curve parametrized by $\vec{r}(t) = \langle t, t^2 \rangle$, where time $t$ is in seconds and the components of $\vec{r}(t)$ are in meters. The temperature at point $(x,y)$ is given by a differentiable function $f(x,y)$ (in degrees Celsius) with gradient $\nabla f(x,y) = \langle x, 2y \rangle$.

(a) At time $t$ find the object's speed, and a unit vector $\vec{u}$ in the direction of the object's motion.

(b) Use your answer to part (a) to estimate approximately how far the object moves between times $t$ and $t + \Delta t$.

(c) At time $t$ find the directional derivative $D_{\vec{u}} f(x,y)$ of the temperature function, at the location $(x,y) = \vec{r}(t)$, in the direction $\vec{u}$ of the object's motion. Your answer should be a function of $t$.

(d) Use your answers to earlier parts to estimate the change in the object's temperature between times $t$ and $t + \Delta t$.

(e) Use your answers to earlier parts to write down a Riemann sum approximating the object's change in temperature between times $t = 0$ and $t = 1$.

(f) Use your answers to earlier parts to write down an integral giving the object's change in temperature between times $t = 0$ and $t = 1$.

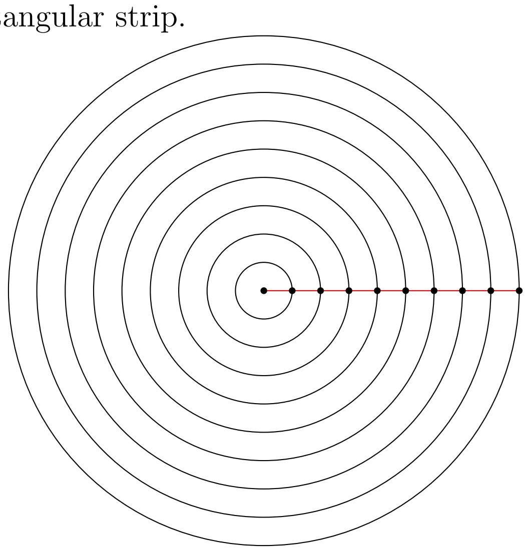
**Solution:** (a) Speed is $|\vec{r}(t)| = \sqrt{1+4t^2}$ m/s, and a unit vector in the direction of motion is $\vec{T}(t) = \frac{\langle 1,2t\rangle}{\sqrt{1+4t^2}}$.
(b) The distance traveled during that time is close to $\sqrt{1+4t^2}\Delta t$ meters.
(c) $D_{\vec{u}} f(x(t), y(t)) = \langle t, 2t^2\rangle \cdot \frac{\langle 1,2t\rangle}{\sqrt{1+4t^2}} = \frac{t+4t^3}{\sqrt{1+4t^2}} = t\sqrt{1+4t^2}$ degrees per meters.
(d) The change in temperature in that time is close to $(t+4t^3)\Delta t$ Celsius degrees.
(e) The change in temperature between times $t = 0$ and $t = 1$ is roughly $\sum_{i=0}^{n}(t_i^* + 4t_i^{*3})\Delta t$ Celsius degrees.

(f) The change in temperature between times $t = 0$ and $t = 1$ is $\int_0^1 t + 4t^3 dt = 1.5$ Celsius degrees.

**Note:** Be sure to include units, and explain your answers. Each part of this problem will be graded based on whether it follows correctly from earlier parts.

13. A thin disc of radius $\pi$ meters is made of material of varying density. At a point $r$ meters from the center of the disc, the mass density per unit area of the disc is $100 + r^2$ grams per square meter. You wish to find the total mass of the disc.

    Begin by dividing up the disc into pieces and estimating the mass of each piece. It will be easiest to make your pieces in the form of thin rings, and consider each ring to be approximately a rectangular strip.



    (a) Approximately what is the area of the $i^{th}$ ring? Be sure to explain your answer.
    (b) Approximately what is the mass of the $i^{th}$ ring? Be sure to explain your answer.
    (c) (Short answer.) Write down a Riemann sum approximating the mass of the disc.
    (d) (Short answer.) Write down an integral giving the mass of the disc.

**Note:** Each part of this problem will be graded based on whether it follows correctly from the earlier parts. If your integral in part (d) does give the mass of the disc, but that integral does not follow correctly from your answer to part (c), then you will not get credit for part (d).

**Solution:** (a)The $i$-th ring is where the distance from the center is roughly $r_i^*$ or $\frac{\pi i}{n}$. Hence, since that ring is similar to a rectangle of length $2\pi r_i^*$ and width $\Delta r = \frac{\pi}{n}$, the area is $2\pi r_i^* \Delta r = 2\frac{\pi^2 i}{n^2}$ meters square.
(b) The mass density is multiplied by the area to give the mass. This is
$2\pi r_i^* \Delta r(100 + r_i^{*2}) = 2\frac{\pi^2 i}{n^2}(100 + (\frac{\pi i}{n})^2)$ grams.
(c)$\sum_{i=0}^{n} 2\pi r_i^* \Delta r(100 + r_i^{*2}) = \sum_{i=0}^{n} 2\frac{\pi^2 i}{n^2}(100 + (\frac{\pi i}{n})^2)$ approximates the mass of the disc, in grams.
(d) $\int_0^\pi 2\pi r(100 + r^2)dr$ is the mass of the disc, in grams.