The Comparison Test and Alternating Series

October 16, 2007

The Comparison Test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

1. If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

2. If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

The Limit Comparison Test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Alternating Series

Definition: An alternating series is a series whose terms are alternately positive and negative. I.e.

$$\sum_{n=1}^{\infty} a_n$$

is alternating if $a_n = (-1)^n b_n$ or $a_n = (-1)^{n-1} b_n$.

The Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

$b_n > 0$ satisfies

1. $b_{n+1} \leq b_n$ for all n

2. $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

Alternating Series Estimation Theorem

If $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

(1) $0 \leq b_{n+1} \leq b_n$ and (2) $\lim_{n\to\infty} b_n = 0$

then $|R_n| = |s - s_n| \leq b_{n+1}$.