Partial Derivatives

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Partial Derivative

Partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation.
Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ then the **partial derivative with respect to** x_i is:

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

we also use f_{x_i} for partial derivative with respect to x_i.

Partial derivative for two variable function

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Rule for taking partial derivatives

To find $\frac{\partial f}{\partial x}$, think of y as a constant and differentiate $f(x, y)$ with respect to x.

Higher order partial derivatives

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Clairaut's Theorem

Suppose that f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{xy} are continuous on D, then

$$\boxed{f_{xy}(a, b) = f_{yx}(a, b)}$$

Tangent Plane

Theorem: If the graph of $f(x, y)$ has a tangent plane at $(a, b, f(a, b))$, then the tangent lane has equation

$$\boxed{L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

Function with partial derivatives and no tangent plane

Example: $f(x, y) = ||x| - |y|| - |x| - |y|$

Differentiable

Definition: A function f is called differentiable at (a, b) if the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ exist **and** if the the function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is a good linear approximation to f near the point (a, b). I.e.

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - L(x, y)}{\|(x, y) - (a, b)\|} = 0$$

Theorem: If the partial derivatives f_x and f_y exist near (a, b) and they are continuous at (a, b) then f is differentiable.

Vector and Matrix notation for $L(x, y)$

$$\begin{aligned} L(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= f(a,b) + \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle x-a, y-b \rangle \\ &= f(a,b) + \begin{pmatrix} f_x(a,b) & f_y(a,b) \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix} \end{aligned}$$