Homework 5 Solution

1. (3.2.17)

$$AB = \begin{bmatrix} 7 & 2+2k \\ 15 & 6+4k \end{bmatrix}$$

$$BA = \begin{bmatrix} 7 & 10 \\ 15 & 6+4k \end{bmatrix}$$

$AB = BA$ if and only if $2+2k = 10$. Hence $AB = BA$ holds only when $\boxed{k=4}$.

2. (3.2.19)

Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$

Then

$$AB = \begin{bmatrix} 3 & 0 \\ -3 & 0 \end{bmatrix} = AC$$

3. (3.2.21)

Let $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Then

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

4. (3.2.25)

a) $\text{Det}(A)\,\text{Det}(B) = (ad-bc)(eh-fg)$

$= adeh - adfg - bceh + bcfg$

b) $\text{Det}(AB) = \det \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

$= (ae+bg)(cf+dh) - (af+bh)(ce+dg)$

$= \cancel{aecf} + aedh + bgcf + \cancel{bgdh}$

$- \cancel{afce} - afdg - bhce - \cancel{bhdg}$

$= aedh + bgcf - afdg - bhce$

c) They are equal, i.e., $\text{Det}(AB) = \text{Det}(A)\,\text{Det}(B)$

5)

a) You can take $u = (x-1)$, $v = y$.

Then

$$\lim_{(x,y)\to(1,0)} \frac{2xy - 2y}{(x-1)^2 + y^2} = \lim_{(u,v)\to(0,0)} \frac{2uv}{u^2 + v^2}.$$

Pick path $C_1 : (t,0)$ $t \to 0$. Then the limit is

$$\lim_{(t,0)\to(0,0)} \frac{2t\cdot(0)}{t^2+0^2} = 0$$

Pick path $C_2 : (t,t)$ $t \to 0$. Then the limit is

$$\lim_{(t,t)\to(0,0)} \frac{2\cdot(t)(t)}{t^2+t^2} = 2.$$

Since the function approach to two different values, the limit does not exist.

b)

Note that

$$\left| \frac{2yx^4}{x^4+y^4} \right| = \frac{x^4}{x^4+y^4} |2y|$$

$$\leq |2y| , \text{ as } \frac{x^4}{x^4+y^4} \leq 1.$$

Hence $$-2|y| \leq \frac{2yx^4}{x^4+y^4} \leq 2|y| .$$

Since both $2|y|$ and $-2|y|$ approach to 0 as $(x,y) \to (0,0)$,

$$\lim_{(x,y)\to(0,0)} \frac{2yx^4}{x^4+y^4} = 0 \text{ by squeeze Thm.}$$

c) Pick $C_1 : (t, 0)$, $t \to 0$. Then the limit is

$$\lim_{(t,0)\to(0,0)} \frac{2 \cdot t \cdot 0 \cdot \cos(t)}{t^2+0} = 0$$

Pick $C_2 : (t^4, t)$, $t \to 0$. Then the limit is

$$\lim_{(t^4,t)\to(0,0)} \frac{2t^4 \cdot t^4 \cos t^4}{t^8+t^8} = \lim_{t\to 0} \cos t^4 = 1$$

Hence the limit does not exist.

6.

North east means that the direction vector $\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

The path is given by $\vec{r}(t) = \langle 2+\frac{1}{\sqrt{2}}t, 1+\frac{1}{\sqrt{2}}t, f(2+\frac{1}{\sqrt{2}}t, 1+\frac{1}{\sqrt{2}}t) \rangle$.

Hence $\vec{r}\,'(0) = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f(2,1) \rangle$.

By def, $D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f(2,1)$

$$= \frac{d}{dt}\bigg|_{t=0} f(2+\tfrac{1}{\sqrt{2}}t, 1+\tfrac{1}{\sqrt{2}}t)$$

$$= \frac{d}{dt}\bigg|_{t=0} (2+\tfrac{1}{\sqrt{2}}t)^2(1+\tfrac{1}{\sqrt{2}}t) + (2+\tfrac{1}{\sqrt{2}}t)(1+\tfrac{1}{\sqrt{2}}t)^3$$

$$= 2(2+\tfrac{1}{\sqrt{2}}t)\cdot\tfrac{1}{\sqrt{2}}\cdot(1+\tfrac{1}{\sqrt{2}}t) + (2+\tfrac{1}{\sqrt{2}}t)^2\tfrac{1}{\sqrt{2}}$$

$$+ \tfrac{1}{\sqrt{2}}(1+\tfrac{1}{\sqrt{2}}t)^3 + (2+\tfrac{1}{\sqrt{2}}t)\cdot 3\cdot(1+\tfrac{1}{\sqrt{2}}t)^2\cdot\tfrac{1}{\sqrt{2}}\bigg|_{t=0}$$

$$= \frac{4}{\sqrt{2}} + \frac{4}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{6}{\sqrt{2}} = \frac{15}{\sqrt{2}} = \boxed{\frac{15}{2}\sqrt{2}}$$

A parallel vector to the tangent line is

$$\vec{r}\,'(0) = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{15}{2}\sqrt{2} \rangle.$$

A simpler parallel vector to the same line is

$$\langle 1, 1, 15 \rangle.$$

Hence parametric eqs of the line is

$$x = 2+t, \quad y = 1+t, \quad z = 6+15t.$$

7.

$$f_x(1,1) = (2x)\Big|_{(x,y)=(1,1)} = 2$$

$$f_y(1,1) = (4y)\Big|_{(x,y)=(1,1)} = 4$$

Hence the tangent plane is $z-3 = 2(x-1) + 4(y-1)$.