Homework 7 Solution

1. (5.6.7)

(a)

$$\text{Proj}_v \langle x,y,z\rangle = \frac{\langle x,y,z\rangle \cdot \langle 1,2,1\rangle}{|\langle 1,2,1\rangle|^2} \cdot \langle 1,2,1\rangle$$

$$= \frac{x+2y+z}{6} \langle 1,2,1\rangle$$

$$= \left\langle \frac{x+2y+z}{6}, \frac{2x+4y+2z}{6}, \frac{x+2y+z}{6} \right\rangle$$

$$[T] = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{1}{6} \\ \frac{2}{6} & \frac{4}{6} & \frac{2}{2} \\ \frac{1}{6} & \frac{2}{6} & \frac{1}{6} \end{bmatrix}$$

$$T \circ T(x) = \text{Proj}_v(\text{Proj}_v(x))$$

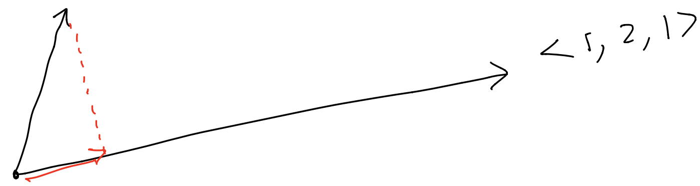
$$= \frac{\frac{x+2y+z}{6} \langle 1,2,1\rangle \cdot \langle 1,3,1\rangle}{|\langle 1,2,1\rangle|^2} \cdot \langle 1,2,1\rangle$$

$$= \frac{x+2y+z}{6} \langle 1,2,1\rangle$$

$$[T \circ T] = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{1}{6} \\ \frac{2}{6} & \frac{4}{6} & \frac{2}{6} \\ \frac{1}{6} & \frac{2}{6} & \frac{1}{6} \end{bmatrix}$$

(b) $[T]$ and $[T \circ T]$ are the same.

(c)



The Projection of any vector onto $\langle 1,2,1\rangle$ is a vector in the same direction as $\langle 1,2,1\rangle$. Since the projection of any vector in the same direction as $\langle 1,2,1\rangle$ is equal to itself, $\text{Proj}_v X = \text{Proj}_v(\text{Proj}_v X)$.

2. (5.6.8)

(a) $S(x,y,z) = \text{Proj}_{\langle 2,-1,0\rangle}\langle x,y,z\rangle$

$$= \frac{\langle x,y,z\rangle \cdot \langle 2,-1,0\rangle}{2^2+(-1)^2+0^2} \cdot \langle 2,-1,0\rangle$$

$$= \frac{2x-y}{5}\langle 2,-1,0\rangle$$

$$[S] = \begin{bmatrix} \frac{4}{5} & \frac{-2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

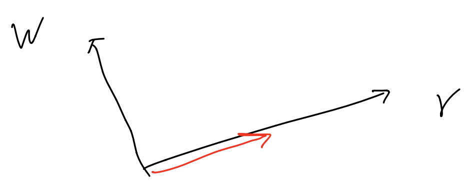
$$[S \circ T] = [S][T]$$

$$= \begin{bmatrix} \frac{4}{5} & \frac{-2}{5} & 0 \\ \frac{-2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{1}{6} \\ \frac{2}{6} & \frac{4}{6} & \frac{2}{6} \\ \frac{1}{6} & \frac{2}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Since $[S \circ T] = [0]$, the linear transformation $S \circ T$ is zero, i.e., $S \circ T(x) = 0$ for all x in $\mathbb{R}^3$.

(c) Note that $v \cdot w = 0$, which means that v is orthogonal to w.



Since $\text{Proj}_v \langle x, y, z \rangle$ is a vector parallel to v, $\text{Proj}_v \langle x, y, z \rangle$ is also orthogonal to w.

Therefore, $\text{Proj}_w \text{Proj}_v \langle x, y, z \rangle$ must be zero.

3.

(a)

If T is a rotation, then $[T] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

for some θ. But that implies $\cos\theta = 1$, $\sin\theta = 1$, which is not possible, as $\sin^2 t + \cos^2 t = 1$ for all t.

(b)

$[Rot_\alpha \circ Rot_\beta]$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

by sum of angle formula

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix},$$ which

is $[Rot_{\alpha+\beta}]$.

(c) We argue that $S \circ L$ is not a rotation. The argument for $L \circ S$ is similar.

Since L is a rotation, then $L = Rot_\alpha$ for some angle α.

Suppose that $L \circ S = Rot_\alpha \circ S$ is a rotation. Then

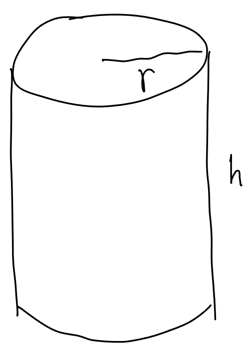
$$L \circ S = Rot_\alpha \circ S = Rot_\beta \quad \text{for some angle } \beta, \text{ which means}$$

$$Rot_{-\alpha} Rot_\alpha S = Rot_{-\alpha} \circ Rot_\beta$$

or

$$S = Rot_{-\alpha} \circ Rot_\beta,$$ which implies that S is a rotation, contradicting to the assumption that S is not a rotation.

4.



Volume of the cylinder $= V(r,h) = \pi r^2 h$

where r = radius, h = height.

Let $r(t)$ = radius at time t,

$h(t)$ = height at time t.

$$\frac{d}{dt} V(r(t), h(t)) = V_r(r(t), h(t)) \frac{dr}{dt} + V_h(r(t), h(t)) \frac{dh}{dt}$$

$$= 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

As $r=5$, $h=10$, $r'=0.5$, $h'=-1$ at the instant,

$$\frac{d}{dt} V(r(t), h(t)) = 2\pi\, 5 \cdot 10 \cdot 0.5 + \pi \cdot 5^2 (-1)$$

$$= 50\pi - 25\pi = 25\pi$$

5. (6.2.2)

(a) $f(x,y,z) = e^{2x-yz}$

$$f'(x,y,z) = [\, f_x \quad f_y \quad f_z \,]$$

$$= [\, 2e^{2x-yz} \quad -z e^{2x-yz} \quad -y e^{2x-yz} \,]$$

$$f'(1,2,1) = [\, 2e^0 \quad -1e^0 \quad -2e^0 \,]$$

$$= [\, 2 \quad -1 \quad -2 \,]$$

(b)
$$L(x,y,z) = f(1,2,1) + [\, 2 \quad -1 \quad -2 \,] \begin{bmatrix} x-1 \\ y-2 \\ z-1 \end{bmatrix}$$

$$= 1 + 2(x-1) - (y-2) - 2(z-1)$$

$$f(0.9, 1.8, 1.1) \approx L(0.9, 1.8, 1.1)$$

$$= 1 + 2(0.9-1) - (1.8-2) - 2(1.1-1)$$

$$= 1 - 0.2 + 0.2 - 0.2$$

$$= \boxed{0.8}$$

(c)

Let $\vec{u} = \dfrac{\langle 1, 2, 3 \rangle}{\sqrt{1^2+2^2+3^2}} = \dfrac{\langle 1, 2, 3 \rangle}{\sqrt{14}}$ = the unit-vector in the same direction.

$D_{\vec{u}} f(1,2,1)$

$$= [f'(1,2,1)] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= [2 \quad -1 \quad -2] \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$$

$$= 2\left(\frac{1}{\sqrt{14}}\right) - \left(\frac{2}{\sqrt{14}}\right) - 2\left(\frac{3}{\sqrt{14}}\right)$$

$$= \boxed{\frac{-6}{\sqrt{14}}}$$