ON THE DENSITY OF ABUNDANT NUMBERS

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Abstract

In [3, 4], Behrend initiates the study of the asymptotic density of abundant numbers. More recently Deléglise [8] used Behrend's upper bound ideas to calculate improved bounds on this density. In this work, we make further improvements to the Deléglise algorithm to determine new bounds on the density of abundant numbers. We will also turn Behrend's lower bound idea into an alternative method of bounding the density.

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Chapter 1

Introduction

1.1 Numbers, perfect and otherwise

From antiquity there has been a fascination with numbers considered perfect, that is, those numbers whose proper divisors sum to the numbers themselves. The sequence of perfect numbers then known was

$$6, \quad 28, \quad 496, \quad 8128,$$

and various assertions were made concerning these numbers, such as:

(a) There are infinitely many perfect numbers.

(b) All perfect numbers are even.

Euclid's proof that if a prime number $p = 2^{n+1} - 1$ for some natural number n then $2^n p$ is perfect, may have bolstered the belief in these statements. In fact, if they had access to the list of currently known perfect numbers, they would have nothing to amend. As of July 2010, 47 perfect numbers were known, all even. (Of course, it can be argued that the computer programs used to find the largest of these were only looking for the even ones.) Nevertheless, the proof of the original claims remain as elusive today as ever. These are among the oldest problems in mathematics. [18]

What about numbers that are not perfect? As early as c. 100 A.D., Nicomachus [9] classified these into two categories: If a number was found to have a sum of aliquot parts which exceeds the number itself, it was considered abundant; if the sum was smaller, it was called deficient. In contrast to the high opinion that the perfect numbers received, the abundant and deficient numbers were often scorned as being of an inferior class. This feeling may be detected in some alternative ways in which the numbers were referred, such as the rather dramatic terms *superfluous* and *defective* for abundant and deficient numbers, respectively. There is even a hint of this remaining in the modern term 'deficient,' which word still has a negative connotation.

Turning our attention to mathematics, a natural (and neutral) question to ask about these numbers is: How many are there of each type? As we have indicated, it is still not known how many perfect numbers there are, whether there are finitely many or infinitely many. But we do know that there are infinitely many numbers in each of the other categories. In fact, we know more: there are more deficient numbers than abundant, in a sense that we will make precise in the next section.

1.2 Natural density

Consider the sequence of natural numbers and the sequence of positive even numbers:

$$1, 2, 3, 4, 5, \ldots, \qquad 2, 4, 6, 8, 10, \ldots.$$

Although we are trained as mathematicians to think of these sets as having the same size in the sense of cardinality, our natural inclination is to think of the second sequence as “half as dense” as the first. Where does this intuition come from? First, we have access to the initial terms of each sequence, and can compare the number of terms up to some bound x. Then we extrapolate, using our imaginations to guess what would happen when x is large. We seek to capture this notion of “density” in the following definition.

Definition 1.1. Let $\mathscr{S}$ denote a subset of the natural numbers and for $x \geq 1$ let $\mathscr{S}(x) = \mathscr{S} \cap [1, x]$ be the set consisting of the elements of $\mathscr{S}$ not exceeding x. We define the *natural density* of $\mathscr{S}$, $\mathbf{d}\,\mathscr{S}$, to be the limit

$$\mathbf{d}\,\mathscr{S} = \lim_{x\to\infty} \frac{|\mathscr{S}(x)|}{x},$$

if such a limit exists. In any case the lim sup and lim inf exist, and these are respectively called the *upper* and *lower natural densities*, with the corresponding notations $\overline{\mathbf{d}}\mathscr{S}$ and $\underline{\mathbf{d}}\mathscr{S}$.

Thus if we let $\mathbb{N}$ be the set of natural numbers, it is easy to see that $\overline{\mathbf{d}}\mathbb{N} = \underline{\mathbf{d}}\mathbb{N} = 1$ so that $\mathbf{d}\,\mathbb{N} = 1$ and likewise defining $\mathscr{E}$ to be the set of even numbers, $\overline{\mathbf{d}}\mathscr{E} = \underline{\mathbf{d}}\mathscr{E} = \frac{1}{2}$ so $\mathbf{d}\,\mathscr{E} = \frac{1}{2}$, as we had anticipated.

It may be the case that a set does not have a natural density. For instance, the following ad hoc construction produces such a set. Put

$$\mathscr{S} = \{1, 4, 5, 6, 7, 16, 17, \dots\},$$

where the numbers in the intervals $[2^n, 2^{n+1})$ are included if n is even, and excluded if n is odd. Calling numbers in the interval $[2^n, 2^{n+1})$ the n-block, we see that since each n-block contains 2^n elements, we find that counting the members of $\mathscr{S}$ up to the $2m$-block gives the bound

$$\overline{\mathbf{d}}\mathscr{S} \geq \lim_{m\to\infty} \frac{|\mathscr{S}(2^{2m+1}-1)|}{2^{2m+1}-1} = \lim_{m\to\infty} \frac{(4^{m+1}-1)/3}{2^{2m+1}-1} = \frac{2}{3},$$

while counting up to the $2m+1$-block gives

$$\underline{\mathbf{d}}\mathscr{S} \leq \lim_{m\to\infty} \frac{|\mathscr{S}(2^{2m+2}-1)|}{2^{2m+2}-1} = \lim_{m\to\infty} \frac{(4^{m+1}-1)/3}{2^{2m+2}-1} = \frac{1}{3}.$$

Since $\underline{\mathbf{d}}\mathscr{S} \neq \overline{\mathbf{d}}\mathscr{S}$, $\mathbf{d}\,\mathscr{S}$ does not exist in this case.

On the other hand, many sequences of numbers which are of interest to number theorists do have a density. Some of these sequences have the general tendency of the terms spreading out as we look at increasingly larger terms, in which case their densities are zero. In this class we find such examples as the sequence of square numbers, that of the prime numbers, and of the set of numbers that are the product of two distinct primes. To see this we first note in general that if we know for a set $\mathscr{S}$ that the number of its members not greater than x, $|\mathscr{S}(x)|$, grows as $O(x \cdot f(x))$ for some function $f(x) = o(x)$, then by the definition of natural density the set will have natural density zero. Among the examples mentioned, the first set is clearly $O(\sqrt{x}) = O(x \cdot x^{-1/2})$, and the second set is $O(x \cdot 1/\log x)$ by the Prime Number Theorem. To estimate the third set, we find an upper bound for the quantity of numbers $pq \leq x$ where p and q are prime and $p < q$. Then $p < \sqrt{x}$, and there are

$\pi(x/p)$ choices possible for q. Thus the number of $pq \le x$ is

$$\sum_{p<\sqrt{x}} \pi(x/p) = O\left(\sum_{p<\sqrt{x}} \frac{x/p}{\log(x/p)}\right) = O\left(\frac{x}{\log x} \sum_{p<\sqrt{x}} \frac{1}{p}\right) = O\left(x \cdot \frac{\log\log x}{\log x}\right).$$

From the foregoing discussion, it becomes apparent that sets with non-trivial (non-zero) density are in some sense special, and we may wonder which sets have this property. Certainly in analogy with the example of the even numbers, we have that any set of multiples has non-trivial density. Rather than proving this directly, we first note that a set of multiples is a special case of a set which is periodic, in the sense that if we listed the members of such a set along the number line and partitioned the natural numbers into intervals $[1, n], [n+1, 2n], \ldots$, each of length n for some $n \in \mathbb{N}$, we would observe the same spacing between numbers in each interval. In other words, the set is some union of congruence classes modulo n.

Definition 1.2. We say that a set $\mathscr{S}$ is *periodic* if it can be written as a union of equivalence classes $a_i \bmod n$ for some $n \in \mathbb{N}$ and $a_i \in [1, n]$,

$$\mathscr{S} = \bigcup_{i=1}^{k} (a_i + n\mathbb{N}_0).$$

Then we say that $\mathscr{S}$ has period n. We call a set *eventually periodic* if it can be written as a union of a finite set and a set which is the translate of a periodic set.

We first prove that any periodic set has a density.

Lemma 1.3. *If $\mathscr{S}$ is periodic with period n, then*

$$\mathbf{d}\,\mathscr{S} = \frac{|\mathscr{S}(n)|}{n}.$$

Proof. Since $\mathscr{S}$ is periodic, we may write it as

$$\mathscr{S} = \bigcup_{a \in A} (a + n\mathbb{N}_0)$$

for some set of natural numbers $A \subseteq [1, n]$. Then

$$\begin{aligned}\mathbf{d}\,\mathscr{S} &= \lim_{x\to\infty} \frac{|\mathscr{S}(x)|}{x} \\ &= \lim_{x\to\infty} \frac{|\sum_{a\in A}(a + n\mathbb{N}_0)(x)|}{x} \\ &= \lim_{x\to\infty} \frac{\sum_{a\in A}\left(\frac{x}{n} + O(1)\right)}{x} \\ &= \frac{|A|}{n} \\ &= \frac{|\mathscr{S}(n)|}{n},\end{aligned}$$

proving the Lemma. □

Noting that any set of multiples is a periodic set, we see that such a set always has a density. Moreover, any finite union of multiple sets is a periodic set, so these also have densities. We now go a step further. Since a density is determined by taking a limit out to infinity, we might hope that any irregularities at the beginning of a sequence would "wash out," so that eventually periodic sets behave in the same way as periodic ones in the limit. We will prove this in the next proposition.

Proposition 1.4. *Let $\mathscr{S}'$ be a finite set, $\mathscr{T}$ a periodic set with period n, and $t \in \mathbb{N}_0$, so that $\mathscr{S} = \mathscr{S}' \cup (t + \mathscr{T})$ is eventually periodic. Then*

$$\mathbf{d}\,\mathscr{S} = \frac{|\mathscr{T}(n)|}{n}.$$

Proof. Let $m = \max(\mathscr{S}' \cup \{t\})$ so that $\mathscr{T} \cap (m, \infty)$ is periodic with period n. Then

$$\begin{aligned}
\mathbf{d}\,\mathscr{S} &= \lim_{x\to\infty} \frac{|\mathscr{S}(m)| + |\mathscr{S} \cap (m,x]|}{x} \\
&= \lim_{x\to\infty} \frac{|\mathscr{S}(m)| + |\mathscr{T} \cap [1,x-m]|}{x} \\
&= \lim_{x\to\infty} \frac{|\mathscr{S}(m)| - |\mathscr{T} \cap [x-m,x]| + |\mathscr{T}(x)|}{x} \\
&= \lim_{x\to\infty} \frac{|\mathscr{T}(x)| + O(1)}{x} \\
&= \frac{|\mathscr{T}(n)|}{n}
\end{aligned}$$

by Lemma 1.3. □

So now we know how to solve the density problem for a fairly large class of sets. Are these the only sets with nontrivial density? Interestingly, there are sets having nontrivial density that are not eventually periodic.

Example 1.5. It is easy to show that the set of squarefree numbers is not eventually periodic. For suppose it were. Then it would contain the set $a + n\mathbb{N}$ for some a, n, and in particular the number $a + a(n+2)n = a(n+1)^2$ would be squarefree, a contradiction. It is known that the density of the set of squarefree numbers is $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$. With this additional information it is immediate that the set of squarefree numbers is not eventually periodic. Indeed, we observe that any eventually periodic set must have a rational number as its density.

Another issue which will arise is the question of infinite additivity. That is, is it true that a density for a set S can be determined by first partitioning it into infinitely many subsets S_i and determining the densities of S_i for each i, and then summing these densities? The following examples will answer this question in the negative.

Example 1.6. Consider the set of natural numbers $\mathbb{N}$. If we partition this set into singleton sets $S_n = \{n\}$ for $n = 1, 2, \ldots$, then it is clear that $\mathbf{d}\, S_n = 0$ for each n, but

$$\mathbf{d} \bigcup_{n=1}^{\infty} \{n\} = \mathbf{d}\, \mathbb{N} = 1 \neq 0 = \sum_{n=1}^{\infty} \mathbf{d}\, S_n.$$

Thus we see that densities are not infinitely additive.

Example 1.7. We return to the example of the set of squarefree numbers. Recall that the density of the primes and the density of squarefree numbers having two prime factors are each zero. It can be shown by induction that the density of squarefree numbers having k primes is zero for any k. If we denote this set by $\mathscr{S}_k$, we conclude that

$$\mathbf{d}\, \mathscr{S} = \mathbf{d} \left(\bigsqcup \mathscr{S}_k \right) = \frac{\pi^2}{6} \neq 0 = \sum_{k=1}^{\infty} \mathbf{d}\, \mathscr{S}_k,$$

so that in this case, as well, densities are not infinitely additive.

1.3 The densities of abundant, perfect, and deficient numbers

Denote the sets of deficient, perfect, and abundant numbers by $\mathscr{D}$, $\mathscr{P}$, and $\mathscr{A}$, respectively. We will often be considering the set of non-deficient numbers, so we also write $\mathscr{A}' = \mathscr{D}^c = \mathscr{P} \cup \mathscr{A}$. Harold Davenport [5], basing his work on Isaac Schoenberg's [31], proved that each of these densities exists and that $\mathbf{d}\, \mathscr{P} = 0$. Note that the complement of an eventually periodic set is eventually periodic. As we will subsequently see that $\mathscr{A}'$ is not eventually periodic, neither is $\mathscr{D}$. Also, since limits are finitely additive, we have $\mathbf{d}(\mathscr{P} \cup \mathscr{A}) = \mathbf{d}\, \mathscr{A}$ and $\mathbf{d}\, \mathscr{D} = 1 - \mathbf{d}\, \mathscr{A}$. This shows that we need only determine one of $\mathbf{d}\,\mathscr{D}$ or $\mathbf{d}\,\mathscr{A}$ to find the other, so in particular we may focus our attention on the natural density of abundant numbers. However, there is no known closed form expression for $\mathbf{d}\,\mathscr{A}$. In 1932, generalizing a method of Issai Schur, Felix Behrend [3] showed that for all n

$$\frac{|\mathscr{A}(n)|}{n} < 0.47.$$

Thus, taking for granted Davenport's result on the existence of the density, there are more deficient numbers in density than there are abundant numbers. In the following year (in fact in the same year as the Davenport density result), Behrend [4] showed for his doctoral dissertation that for large n

$$0.241 < \frac{|\mathscr{A}(n)|}{n} < 0.314,$$

so that there are at least twice as many deficients as abundants. These bounds were later improved by Hans Salié [30] ($0.246 < \mathbf{d}\,\mathscr{A}$) Charles Wall, et al., [35, 36] ($0.2441 < \mathbf{d}\,\mathscr{A} < 0.2909$, note that the lower bound is worse than Saliè's!) and finally by Marc Deléglise [8] who found the current bounds

$$0.2474 < \mathbf{d}\,\mathscr{A} < 0.2480,$$

giving $\mathbf{d}\,\mathscr{A} = 0.247\ldots$ so that the density of abundant numbers is slightly less than 1/4.

At this point a number of questions naturally arise. Noting the painfully slow progress made until now in tightening the bounds for $\mathbf{d}\,\mathscr{A}$, we may wonder how much these results can be significantly improved. For instance, can the next digit be found? If the bounds cannot be significantly improved, why not? Finally, can the study of $\mathscr{A}$ be generalized to a larger class of sets? In the remainder of this work we will begin to answer such questions by continuing where Deléglise's paper left off and study improvements in the calculation of the density of abundant numbers as well as other related sets.

1.4 Outline of work

We will begin our study by returning to the original work of Behrend. With the Erdős–Wintner Theorem at our disposal, we will be able to recast his work in terms of densities. This will lead us to the method used by Deléglise to bounding the density of abundant numbers. Deléglise took the upper bound method used by Behrend and wrote a program that can calculate both upper and lower bounds for the density of abundant numbers.

Our first contribution will be to study the computational complexity of the program used by Deléglise. Here a special role is played by the numbers $n \le z$ with prime factors of n not exceeding y, and so the counting function of these numbers, $\Psi(z, y)$, makes an appearance. As a consequence, we have y and z as parameters for the Deléglise program. By studying the running time $T(z, y)$ of the program, we find that

$$T(z, y) = O((\log z)^2 \Psi(z, y)).$$

The function $\Psi(z, y)$ has been extensively studied, and it has been found that, taking $y = z^{1/u}$, the behavior of $\Psi(z, z^{1/u})$ is governed by u. Writing the difference between the upper and lower bounds of the Deléglise program as $E(z, y)$, we use results on $\Psi(z, z^{1/u})$ to prove the following.

Theorem 1.8. *With parameters y, z chosen to be $z = y^{\frac{\log\log y}{\log\log\log y}}$ so that*

$$u = \frac{\log\log y}{\log\log\log y},$$

we have

$$E(z, y) \ll \frac{1}{u^u}.$$

This means that the Deléglise program can compute the density of abundant numbers to any desired precision. Combining the results for $T(z, y)$ and $E(z, y)$, we conclude that the running time of the program grows at worst double-exponentially with the number of digits desired in the density:

Corollary 1.9. *Let t be the time that the Deléglise algorithm takes to determine the density $\mathbf{d}\,\mathscr{A}_\alpha$ to within 10^{-k}. Then we have that*

$$t < e^{e^{ck}},$$

where c is an absolute constant.

Next, we will detail a number of improvements that can be made to the Deléglise algorithm in order to close the gap between the upper and lower bounds. The lower bound improvements involve focusing on either primes smaller or larger than y, and are thus called the small primes and large primes methods, respectively. Many of the upper bound improvements involve refining a method of Behrend to bound density using moments of functions related to $\sigma(n)/n$. A further improvement is made by using an asymptotic result of Paul Erdős with explicit constants. By combining all such improvements, we tighten the known bounds to

$$0.2476171 < \mathbf{d}\mathscr{A} < 0.2476475,$$

and thus determine the next decimal digit for the density of abundant numbers; we now know that $\mathbf{d}\mathscr{A} = 0.2476\ldots$.

Returning our attention to Behrend's dissertation, we develop the seed of his lower bound idea. Behrend used a finite subset of primitive nondeficient numbers (pnd's) to determine the density of a subset of abundant numbers. A primitive nondeficient number is a nondeficient number that does not have any nondeficient proper divisors. We also consider the generalized version of these numbers, called α-primitive nondeficient (α-pnd), which are numbers a that have $\sigma(a)/a \geq \alpha$ but proper divisors d with $\sigma(d)/d < \alpha$. By considering the set of all pnd's $\mathbb{P} = \{a_1, a_2, \ldots\}$, which can be used to generate all nondeficient numbers, we show how these can be used to determine the density of the set of the abundant numbers themselves. This is done by first discovering a way of organizing the nondeficient numbers. We are then naturally led to a new infinite series expression for the density of abundant numbers.

Theorem 1.10. *The density of abundant numbers can be expressed as the infinite sum*

$$\mathbf{d}\mathscr{A} = \sum_{a_i \in \mathbb{P}} \frac{\varphi(c_i)}{c_i} \frac{1}{a_i},$$

where $c_i = L_k/a_i$.

Here L_k is the lcm of the first k prime powers ordered in a specific manner. This expression allows us to calculate bounds for the density of abundants. The computational complexity of this new method is examined. All of these results generalize to the α-pnd case.

We conclude with a demonstration of the utility of the new approach to α-pnd's. Some results on α-pnd's are proven, culminating in a new proof of the result of Shapiro that there are infinitely many α-pnd's n with k distinct prime factors if and only if α can be expressed as

$$\alpha = \frac{\sigma(a)}{a} \cdot \frac{b}{\varphi(b)}, \qquad (a, b) = 1, \qquad b > 1,$$

and $\omega(a) + \omega(b) < k$.

Chapter 2

Behrend's thesis

In this chapter, we will introduce the work of Behrend [4], who published the first bounds on the upper and lower density of abundant numbers. In order to simplify our discussion, we will first prove the existence of the density of abundant numbers using the Erdős-Wintner theorem [16]. Next we will describe Behrend's method as explained in general form in Deléglise [8]. This will lay the foundation for our discussion of subsequent chapters when we study the Deléglise algorithm and make improvements in the algorithm.

2.1 Preliminaries

In this section, we establish the basic notation and concepts that will be used in the remainder of this document.

We will always use n and m to mean natural numbers, and use p and q for prime numbers. We denote the product of the primes $p \leq y$ by $\Pi(y)$, with $\Pi(y) = 1$ when $y < 2$. Thus $\Pi(2) = 2$, $\Pi(\pi) = 6$, and $\Pi(5) = 30$. We let $P(n)$ and $p(n)$ denote respectively the largest and smallest prime dividing n when $n > 1$, and $P(1) = p(1) = 1$.

Let n have the canonical prime decomposition

$$n = \prod_{i=1}^{k} p_i^{e_i},$$

namely with p_i primes and $p_i \neq p_j$ when $i \neq j$. We use the notation $m \| n$ to indicate that m is a unitary divisor of n. In other words, m is a divisor of n such that $(m, n/m) = 1$. In particular $1 \| n$ for all n and $p_i^{e_i} \| n$ for each i, $1 \leq i \leq k$.

Proposition 2.1. *Let f be a multiplicative function and $f(p^i) \geq f(p^{i-1})$ on primes p when $i \geq 1$. If m, n are natural numbers, then $f(mn) \geq f(n)$. If the inequality on prime powers is strict, namely $f(p^i) > f(p^{i-1})$ when $i \geq 1$, we have $f(mn) = f(n)$ only in the case $m = 1$.*

Proof. The result is true when $m = 1$. If $m > 1$, let $p^e \| mn$. Then $p^{e'} \| n$ where $0 \leq e' \leq e$. Thus $f(p^e) \geq f(p^{e'})$ for each unitary prime power divisor of mn. Since f is multiplicative, $f(1) = 1$, and with the condition $f(p^i) \geq f(p^{i-1})$ we have for each $i \geq 1$ that $f(p^i) \geq 1$. Then multiplying together the inequalities yields

$$f(mn) = \prod_{p^e \| mn} f(p^e) \geq \prod_{p^{e'} \| n} f(p^{e'}) = f(n).$$

If we have $f(p^i) > f(p^{i-1})$, repeating the argument with this condition results in the strict inequality. □

Example 2.2. We denote by $\sigma(n)$ the sum of the positive divisors of n. Thus $\sigma(1) = 1$, $\sigma(p) = p + 1$, and $\sigma(6) = 1 + 2 + 3 + 6 = 12$. We further define $h(n) = \sigma(n)/n$. It is immediate that

$$h(n) = \sum_{d|n} \frac{1}{d}. \tag{2.1}$$

Since both $\sigma(n)$ and $1/n$ are multiplicative functions, their product $h(n)$ is also multiplicative. We also note that $h(p^i) = h(p^{i-1}) + 1/p^i$. Since h satisfies the hypotheses of the previous proposition, we have that proper multiples of a natural number n have values of h strictly larger than $h(n)$.

In fact, from (2.1) it is easy to see that $h(mn) \geq h(n)$ with equality only when $m = 1$, since the terms of the sum for $h(n)$ is a subset of the terms of the sum for $h(mn)$, and is a proper subset unless $m = 1$. We will be making essential use of this property of h in what follows.

The following example should be compared with the example above.

Example 2.3. We denote by $\varphi(n)$ the Euler φ-function, which is the number of $m \in [1, n]$ which are relatively prime to n. Noting that $\varphi(n)$ is multiplicative, we have that the quotient $n/\varphi(n)$ is multiplicative. In contrast to the previous example, however, we have $p^i/\varphi(p^i) = 1 + 1/(p-1)$ for all $i \geq 1$. Thus we have only that $n/\varphi(n) \leq mn/\varphi(mn)$ when $m > 1$.

2.2 The existence of the density of α-abundants

In 1928, Schoenberg [31] proved for each $\alpha \in [0, 1]$ the existence of the density of numbers n such that $\varphi(n)/n \geq \alpha$. His proof is technical and involves the study of the mean values of the powers of $\varphi(n)/n$, the so-called moments of $\varphi(n)/n$. In 1933, Davenport [5] adapted this proof to work also for $h(n) = \sigma(n)/n$. Incidentally, Davenport reports in the same article that each of Behrend and Sarvadaman Chowla independently proved the same result. In order to state the theorem, we will make the following definitions.

Definition 2.4. Let $\mathscr{D}_\alpha$, $\mathscr{P}_\alpha$, and $\mathscr{A}_\alpha$ denote the sets of numbers n such that $h(n) < \alpha$, $h(n) = \alpha$, and $h(n) > \alpha$, respectively. In addition, we define

$$\mathscr{D}'_\alpha := \mathscr{D}_\alpha \cup \mathscr{P}_\alpha \qquad \text{and} \qquad \mathscr{A}'_\alpha := \mathscr{A}_\alpha \cup \mathscr{P}_\alpha.$$

Then we may state the theorem of Davenport as follows.

Theorem 2.5 (Davenport, et al., 1933)**.** *For each α, $\mathbf{d}\,\mathscr{A}'_\alpha$ exists. Considered as a function in α, $\mathbf{d}\,\mathscr{A}'_\alpha$ is continuous.*

In 1939 a much more general result was proven by Erdős and Wintner [16]. The theorem is stated in terms of a limiting distribution function for an additive arithmetic function. We first define a *distribution function* (d.f.) to be a non-decreasing function $D\colon \mathbb{R} \to [0,1]$ which is right-continuous and satisfies $\lim_{\alpha\to-\infty} D(\alpha) = 0$, $\lim_{\alpha\to\infty} D(\alpha) = 1$. A particularly simple class of distribution functions are those that are step functions. Such a d.f. is called *purely discrete.* These d.f.'s are necessarily not continuous. A simple example of a continuous distribution function is a function that can be defined with a Lebesgue-integrable function $f \geq 0$ such that $\|f\|_1 = 1$, that is, having L^1 norm 1, by

$$D(\alpha) = \int_{-\infty}^{\alpha} f(t)dt.$$

Such a d.f. is called *absolutely continuous.* If a continuous distribution function D

has

$$\int_S dD(\alpha) = 1$$

where $S \subseteq \mathbb{R}$ has zero Lebesgue measure, we say that D is *purely singular*. This type of d.f. is interesting in view of the Radon–Nikodym Theorem which implies that any continuous d.f. is a linear combination of an absolutely continuous d.f. and a purely singular one. If a distribution function is either purely discrete, or continuous and purely singular, or absolutely continuous, we say that the d.f. is of *pure type*.

We next define another function which will turn out to be a distribution function, based on a real-valued arithmetic function f. For each $N \geq 1$ we define the function

$$D_N(\alpha) = D_{N,f}(\alpha) := \frac{1}{N}|\{n \leq N : f(n) \leq \alpha\}|$$

to be a *distribution function for* f. Note that this is indeed a distribution function, and is in fact an example of one that is purely discrete. A sequence $\{F_N\}_{N=1}^{\infty}$ of distribution functions is said to *converge weakly* to a distribution function F if

$$\lim_{N\to\infty} F_N(\alpha) = F(\alpha)$$

at every point α at which F is continuous. If $\{D_N\}_{N=1}^{\infty}$ is a sequence of distribution functions for an arithmetic function f that converges weakly to a distribution function D, we say that f *has a limiting distribution function* D (or simply has a distribution function D, or has a limit law with d.f. D). Now we may state the Erdős–Wintner theorem.

Theorem 2.6 (Erdős, Wintner)**.** *Let f be a real additive function. For f to have a* *limiting distribution function, it is necessary and sufficient that the three series*

$$\sum_{|f(p)|>1} \frac{1}{p}, \qquad \sum_{|f(p)|\le 1} \frac{f(p)}{p}, \qquad \sum_{|f(p)|\le 1} \frac{f^2(p)}{p}$$

converge. The limiting distribution is of pure type. It is continuous if and only if the series

$$\sum_{f(p)\neq 0} \frac{1}{p}$$

diverges.

The Erdős–Wintner Theorem provides the principal tool that we can use to prove the existence of the densities of a large class of sets which includes the set of abundant numbers.

In our first application of Erdős–Wintner, we will prove the Davenport result that for any α, the density of the set $\mathscr{A}'_\alpha$ of numbers n with $\sigma(n)/n \ge \alpha$ exists and varies continuously with α, by showing that the d.f. for $h(n) = \sigma(n)/n$ exists and is continuous. Note that as h is multiplicative, $\log h$ is additive. Thus the Erdős–Wintner theorem will give a result about the distribution function

$$D(\log\alpha) = \lim_{N\to\infty} \frac{1}{N}|\{n \le N : \log h(n) \le \log\alpha\}|$$

for $\alpha > 0$. Note the reversal in the direction of the inequality compared to the sets as stated in the Davenport theorem. In fact, $D(\log\alpha) = \mathbf{d}\,\mathscr{D}'_\alpha$ for $\alpha > 0$. To prove the existence of these densities, we need only check that $\log h$ satisfies the conditions in the Erdős–Wintner theorem. We do this by noting that since $\log h(p) = \log(1+1/p) <$

$1/p < 1$, the three sums in the statement of the theorem are bounded above by

$$\sum_{|\log h(p)|>1} \frac{1}{p} = 0, \qquad \sum_{|\log h(p)|\leq 1} \frac{\log h(p)}{p} \leq \sum_{p} \frac{1}{p^2}, \qquad \sum_{|\log h(p)|\leq 1} \frac{(\log h(p))^2}{p} \leq \sum_{p} \frac{1}{p^3},$$

so each of these sums converges. This proves that $\mathbf{d}\,\mathscr{D}'_\alpha$ exists for $\alpha > 0$. Of course, we have trivially that $\mathbf{d}\,\mathscr{D}'_\alpha = 0$ for $\alpha \leq 0$, so $\mathbf{d}\,\mathscr{D}'_\alpha$ exists for all α. Next, by finite additivity of densities, the density of the complement of $\mathscr{D}'_\alpha$, namely $\mathscr{A}_\alpha$, exists for all α.

We can also prove that the distribution function $D(\log\alpha)$ is continuous. We first note that $\log h(p) \neq 0$ for all p. Then

$$\sum_{\log h(p)\neq 0} \frac{1}{p} = \sum_{p} \frac{1}{p}$$

which diverges by a theorem of Euler [27, p. 7], so $\mathbf{d}\,\mathscr{D}'_\alpha = D(\log\alpha)$ is continuous for $\alpha > 0$. Since $\mathbf{d}\,\mathscr{D}'_\alpha = 0$ for $\alpha \leq 0$, we can again extend continuity of $\mathbf{d}\,\mathscr{D}'_\alpha$ to all α.

We now use this result to show that $\mathbf{d}\,\mathscr{P}_\alpha = 0$. First we note that $\mathscr{P}_\alpha = \mathscr{D}'_\alpha \setminus \mathscr{D}_\alpha$. Now for $\epsilon > 0$ we have that

$$\overline{\mathbf{d}}\left(\mathscr{D}'_\alpha \setminus \mathscr{D}_\alpha\right) \leq \lim_{\epsilon\to 0^+} \mathbf{d}\left(\mathscr{D}'_\alpha \setminus \mathscr{D}'_{\alpha-\epsilon}\right) = \lim_{\epsilon\to 0^+} \left(\mathbf{d}\,\mathscr{D}'_\alpha - \mathbf{d}\,\mathscr{D}'_{\alpha-\epsilon}\right) = 0.$$

We thus conclude that the density of any set $\mathscr{P}_\alpha = \mathscr{D}'_\alpha \setminus \mathscr{D}_\alpha$ is 0, and also that each set $\mathscr{D}_\alpha$ and $\mathscr{A}_\alpha$ has a density. Returning our attention to the case $\alpha = 2$, we conclude that $\mathscr{D}$, $\mathscr{P}$, and $\mathscr{A}$ each have densities, and additionally that $\mathbf{d}\,\mathscr{P} = 0$.

Finally, we make note of another property of the distribution function $\mathbf{d}\,\mathscr{D}'_\alpha$, that for $\alpha > 1$, $\mathbf{d}\,\mathscr{D}'_\alpha$ is strictly increasing. Let $1 \leq \alpha_1 < \alpha_2$. By finite additivity of the density, it suffices to find a set of numbers nm with $\alpha_1 < h(nm) < \alpha_2$ having nonzero density.

We first identify a number n such that

$$\alpha_1 \leq h(n) < h(n)(1+\epsilon) < \alpha_2 \tag{2.2}$$

for some $\epsilon > 0$, and then show that there exists a set of numbers m such that $1 < h(m) \leq 1+\epsilon$ with nonzero density d. Since

$$h(n) \leq h(nm) \leq h(n)h(m),$$

we have the set inclusion

$$\mathscr{S} = \{m : h(nm) \in (h(n), h(n)(1+\epsilon)]\} \supseteq \{m : h(n)h(m) \in (h(n), h(n)(1+\epsilon)]\} = \mathscr{D}'_{1+\epsilon}$$

for any $\epsilon > 0$. We let $d = \mathbf{d}\,\mathscr{D}'_{1+\epsilon}$. Then the density of the set $n\mathscr{S}$ will be bounded below by d/n.

To show that an n satisfying (2.2) exists, we will again use Euler's result that

$$\sum_p \frac{1}{p}$$

diverges, where the sum is over all primes p. Then we see that

$$\prod_p \left(1 + \frac{1}{p}\right)$$

diverges by taking the logarithm and using the bound $cx \leq \log(1+x)$ for $0 \leq x \leq 2$

and $c \leq \log(1+1/2)/2$. This bound can be seen by comparing the graphs of $\log(1+x)$ and cx. Since for each prime p we have $h(p) = 1 + 1/p$, by the multiplicativity of h we can find some finite product of primes $n = \Pi p$ such that

$$\alpha_1 \leq h\left(\prod p\right) = \prod h(p) = \prod \left(1 + \frac{1}{p}\right) < \alpha_2.$$

This will be our desired n.

We will now determine a set of appropriate m. Fix an $\epsilon > 0$ satisfying

$$h(n)(1+\epsilon) < \alpha_2.$$

Now we must establish that $\mathbf{d}\,\mathscr{D}_{1+\epsilon} > 0$. We will use a theorem of Erdős [13].

Theorem 2.7. *As* $\epsilon \to 0^+$,

$$\mathbf{d}\,\mathscr{D}_{1+\epsilon} = (1+o(1))\frac{e^{-\gamma}}{\log \epsilon^{-1}},$$

where γ *is the Euler-Mascheroni constant.*

This establishes our claim.

2.3 Behrend's thesis

The doctoral dissertation of Felix Behrend, published as [4], describes two methods of bounding the density of abundant numbers. For the lower bound, Behrend identifies a set A of 22 nondeficient numbers, from which he calculates the density $\mathbf{d}\,\mathscr{M}(A)$ of the multiples of members of A. Recall that $\mathscr{A} = \mathscr{A}_2$, $\mathscr{A}' = \mathscr{A}_2'$, for the function h.

Since the union of these multiple sets is a subset of $\mathscr{A}'$, we have the lower bound $\mathbf{d}\,\mathscr{M}(A) \leq \mathbf{d}\,\mathscr{A}'$. Since we have shown that $\mathbf{d}\,\mathscr{A}' = \mathbf{d}\,\mathscr{A}$, we have a lower bound for $\mathbf{d}\,\mathscr{A}$, as desired. We will return to this method in more detail later.

Behrend's upper bound method uses two main ideas. One idea is to partition the set of non-deficient numbers according to their smallest prime factors. In Behrend's original paper, the small primes were limited to those up to 7, but Deléglise [8] shows how this can be generalized to primes $\leq y$ for any y. In addition, since at the time the existence of the density was not known, Behrend was confined to work with the upper and lower densities, rather than the density itself, in his argument. We will follow Deléglise and make use of the existence of the density as was proven in the previous section. We will also need the existence of the densities of certain subsets of $\mathscr{A}'$ that we will define below.

Definition 2.8. Suppose we factor a number $n = uv$ so that the prime factors of u are at most y and the prime factors of v are greater than y. We will call u the *y-smooth part* of n. In addition, we call a number n *y-smooth* if every prime factor p of n is less than or equal to y, namely if n is its own y-smooth part.

Denote by $\mathscr{A}_y^n$ the set of non-deficient numbers that have y-smooth part n. Thus

$$\mathscr{A}' = \bigsqcup_{P(n) \leq y} \mathscr{A}_y^n,$$

where the disjoint union is over all y-smooth numbers n, and $P(n)$ is the largest prime dividing n. We first ask whether each $\mathscr{A}_y^n$ has a density. If n is abundant, this is easy since $\mathscr{A}_y^n$ is then the set of multiples mn of n with $(m, \Pi(y)) = 1$, which is a periodic set, so by Lemma 1.3, $\mathscr{A}_y^n$ has a density.

In the case that n is deficient, we do not have a periodic set, so we use the Erdős–Wintner theorem with the following arithmetic function.

Definition 2.9. Let h_y be the multiplicative function defined on prime powers p^e as

$$h_y(p^e) = \begin{cases} 1, & p \leq y, \\ h(p^e), & p > y, \end{cases}$$

where $h(n) = \sigma(n)/n$. We also define the sets

$$\mathscr{H}_{y,\alpha} = \{m : h_y(m) \geq \alpha\}$$

and

$$\mathscr{A}^n_{y,\alpha} = \{m \in \mathscr{H}_{y,\alpha} : h_y(m) \geq \alpha, y\text{-smooth part of } m \text{ is } n\}.$$

Note that $\mathscr{A}^n_{y,\alpha}$ is a generalization of the set $\mathscr{A}^n_y$ since $\mathscr{A}^n_y = \mathscr{A}^n_{y,2/h(n)}$. To see this, we take $mn \in \mathscr{A}^n_y$, where m contains only primes greater than y. Since $(m,n) = 1$, we have

$$h(mn) = h(m)h(n) \geq 2 \iff h(m) \geq \frac{2}{h(n)}.$$

With these definitions, it is clear that the Erdős–Wintner theorem applies in our situation. The distribution function for the arithmetic function h_y exists since it exists for the related function h, and $h_y(p) \neq h(p)$ on only finitely many primes, so it does not affect the convergence properties of the three series to be checked in the Erdős–Wintner theorem. We conclude that $\mathbf{d}\,\mathscr{A}^n_{y,\alpha}$ exists for α, and in particular $\mathbf{d}\,\mathscr{A}^n_y$ exists.

Next we would like to express the density of the set of abundant numbers $\mathscr{A}$ in terms of the densities of its subsets $\mathscr{A}_y^n$. As we have seen, densities are not infinitely additive, so it is not immediately clear that summing the densities of $\mathscr{A}_y^n$ over the infinitely many y-smooth numbers n will give us the density of the union of the $\mathscr{A}_y^n$. Nevertheless, we will be able to prove that this is the case. First we will need a lemma.

Lemma 2.10. *The sum of reciprocals of the y-smooth numbers n is*

$$\sum_{P(n)\le y} \frac{1}{n} = \prod_{p\le y}\left(1-\frac{1}{p}\right)^{-1}.$$

Proof. Define $e_p = \lfloor \log x/\log p\rfloor$ so that e_p is the largest integer exponent such that $p^{e_p} \le x$. Then

$$\sum_{\substack{n\le x\\ P(n)\le y}} \frac{1}{n} \le \prod_{p\le y}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{e_p}}\right) \le \sum_{\substack{n\le x^{\pi(y)}\\ P(n)\le y}} \frac{1}{n}.$$

Then taking limits as $x\to\infty$, we have our result. □

In view of the foregoing, we will introduce the notation

$$F(y) := \prod_{p\le y}\left(1-\frac{1}{p}\right) = \frac{\varphi(\Pi(y))}{\Pi(y)}.$$

We can now prove the following proposition.

Proposition 2.11. *Let $\mathscr{A}$ be the set of abundant numbers and let $\mathscr{A}_y^n$ be the set of nondeficient numbers with y-smooth part n. Then*

$$\mathbf{d}\,\mathscr{A} = \sum_{P(n)\le y} \mathbf{d}\,\mathscr{A}_y^n$$

where the sum is over all y-smooth numbers n.

Proof. Recall from the discussion at the end of Section 2.2 that $\mathbf{d}\,\mathscr{A}' = \mathbf{d}\,\mathscr{A}$. We will thus prove the statement using the set $\mathscr{A}'$ of nondeficient numbers. We split the disjoint union according to whether $n \leq z$ or $n > z$ so

$$\mathscr{A}' = \left(\bigsqcup_{\substack{n \leq z \\ P(n) \leq y}} \mathscr{A}_y^n \right) \sqcup \left(\bigsqcup_{\substack{n > z \\ P(n) \leq y}} \mathscr{A}_y^n \right).$$

Using that $\mathscr{A}'$ and each $\mathscr{A}_y^n$ have densities, the union $\sqcup_{n>z, P(n) \leq y} \mathscr{A}_y^n$ also has a density since its complement in $\mathscr{A}'$ has a density. Thus

$$\mathbf{d}\,\mathscr{A}' = \sum_{\substack{n \leq z \\ P(n) \leq y}} \mathbf{d}\,\mathscr{A}_y^n + \mathbf{d} \left(\bigsqcup_{\substack{n > z \\ P(n) \leq y}} \mathscr{A}_y^n \right).$$

We must show that the final term goes to zero as $z \to \infty$. Since $\mathscr{A}_y^n \subseteq n\mathbb{N}$,

$$|\mathscr{A}_y^n(x)| \leq |n\mathbb{N}(x)| \leq \frac{x}{n}.$$

By the subadditivity of $\limsup$,

$$\overline{\mathbf{d}} \left(\bigsqcup_{\substack{n > z \\ P(n) \leq y}} \mathscr{A}_y^n \right) \leq \sum_{\substack{n > z \\ P(n) \leq y}} \overline{\mathbf{d}} \mathscr{A}_y^n \leq \sum_{\substack{n > z \\ P(n) \leq y}} \frac{1}{n}.$$

By the previous lemma we know that the final expression is the tail of a convergent series. Thus the tail goes to 0 as $z \to \infty$ and we have proven our result. □

This proposition allows us to reduce determining the density of $\mathscr{A}$ to determining the density of $\mathscr{A}_y^n$ for each y-smooth n.

Finally, we express the sets $\mathscr{A}_y^n$ in a way that shows explicitly the property that all of its members are multiples of n.

Definition 2.12. Let $\mathscr{A}_{y,\alpha}$ denote the set of numbers m such that $(m, \Pi(y)) = 1$ and $h(m) \geq \alpha$.

Then $\mathscr{A}_{y,\alpha}^n$ is related to $\mathscr{A}_{y,\alpha}$ by

$$\mathscr{A}_{y,\alpha}^n = n\mathscr{A}_{y,\alpha},$$

and their densities are related by

$$\mathbf{d}\,\mathscr{A}_{y,\alpha}^n = \frac{\mathbf{d}\,\mathscr{A}_{y,\alpha}}{n}. \tag{2.3}$$

With Equation (2.3) and the relationship between $\mathscr{A}_{y,\alpha}^n$ and $\mathscr{A}^n$, Proposition 2.11 can be written

$$\mathbf{d}\,\mathscr{A} = \sum_{P(n)\leq y} \frac{\mathbf{d}\,\mathscr{A}_{y,2/h(n)}}{n}.$$

Also note that this result does not rely on the bound $h(n) \geq 2$ in an essential way. Thus we have in fact proved for any α that

$$\mathbf{d}\,\mathscr{A}_\alpha = \sum_{P(n)\leq y} \frac{\mathbf{d}\,\mathscr{A}_{y,\alpha/h(n)}}{n}. \tag{2.4}$$

We now move on to the second idea of Behrend, a primitive form of which he credits to Schur in [3]. Behrend was able to find upper bounds for densities of $\mathscr{A}_{y,\alpha}$ using the moments of h, which we define as follows.

Definition 2.13. For an arithmetic function f, we define the *mean* of f to be

$$M(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n),$$

and the *rth moment of* f to be

$$M_r(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f^r(n),$$

if these limits exist.

Note that the rth moment of f is simply the mean of f^r, so any result on means holds equally well for moments.

Next we will use the following proposition.

Proposition 2.14. *Let f be an arithmetic function such that for some $\alpha_0 \ge 0$, we have $f(n) \ge \alpha_0$ for all natural numbers n, and for some $\alpha > \alpha_0$, let $\mathscr{N}$ denote the set of n such that $f(n) \ge \alpha$. Suppose that both the mean of f and the density of $\mathscr{N}$ exist. Then*

$$\mathbf{d}\,\mathscr{N} \le \frac{M(f) - \alpha_0}{\alpha - \alpha_0}.$$

Proof. We observe that

$$\begin{aligned} M(f) &= \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) \\ &= \lim_{x\to\infty} \frac{1}{x} \left(\sum_{\substack{n\le x \\ f(n)<\alpha}} f(n) + \sum_{\substack{n\le x \\ f(n)\ge\alpha}} f(n) \right) \\ &\ge \lim_{x\to\infty} \frac{1}{x} \left(\alpha_0(\lfloor x \rfloor - |\mathscr{N}(x)|) + \alpha|\mathscr{N}(x)|\right) \end{aligned}$$

$$= (\alpha - \alpha_0)\,\mathbf{d}\,\mathscr{N} + \alpha_0.$$

Solving for $\mathbf{d}\,\mathscr{N}$, we arrive at our result. □

We wish to apply this proposition to our sets $\mathscr{A}^n_{y,\alpha}$. Thus, we must prove that the moments of h_y exist. Let μ be the Möbius function. For two arithmetic functions f and g, the Möbius inversion formula [1, p. 32] gives that

$$f(n) = \sum_{d|n} g(d) \qquad \Longleftrightarrow \qquad g(n) = \sum_{d|n} f(d)\mu\left(\frac{n}{d}\right),$$

and we say that g is the Möbius inverse of f. Writing

$$h'(n) = \sum_{d|n} h^r_y(d)\mu\left(\frac{n}{d}\right)$$

so that h' is the Möbius inverse of h^r_y, we have

$$\begin{aligned}
M(h^r_y) &= \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} h^r_y(n) \\
&= \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \sum_{d|n} h'(d) \\
&= \lim_{x\to\infty} \frac{1}{x} \sum_{d\le x} h'(d) \left\lfloor \frac{x}{d} \right\rfloor \\
&= \lim_{x\to\infty} \frac{1}{x} \left(\sum_{d=1}^{\infty} h'(d)\frac{x}{d} + E(x) \right),
\end{aligned}$$

where

$$E(x) = \sum_{d\le x} h'(d) \left(\left\lfloor \frac{x}{d} \right\rfloor - \frac{x}{d} \right).$$

We will show that

$$|E(x)| \leq \sum_{n \leq x} |h'(n)| = o(x)$$

as $x \to \infty$ so that

$$M(h_y^r) = \sum_{d=1}^{\infty} \frac{h'(d)}{d},$$

and we will show that this last series converges.

First we note that h' is positive. By definition we have

$$h'(p^e) = h_y^r(p^e) - h_y^r(p^{e-1})$$

and $h_y(p^e) \geq h_y(p^{e-1}) \geq 1$, so that h' is non-negative on prime powers $p^e, e \geq 1$. Thus, h' is always non-negative. Using this, we can bound $|E(x)|$ by

$$0 \leq -E(x) \leq \sum_{n \leq x} h'(n) \leq \prod_{p \leq x} \left(1 + h'(p) + h'(p^2) + \cdots + h'(p^{e_p})\right) = \prod_{p \leq x} h_y^r(p^{e_p}),$$

where as before $e_p = \lfloor \log x / \log p \rfloor$ is the largest number so that $p^{e_p} \leq x$.

Next we estimate $h_y^r(p^{e_p})$. We have

$$1 \leq h_y^r(p^{e_p}) < \left(1 + \frac{1}{p-1}\right)^r = 1 + O_r\left(\frac{1}{p}\right) \tag{2.5}$$

where the constant implied by the big-O depends only on r uniformly over p and e_p. This leads us to the product

$$\prod_{p \leq x} h_y^r(p^{e_p}) = \exp \sum_{p \leq x} \log\left(1 + O_r\left(\frac{1}{p}\right)\right) \leq \exp\left(O_r\left(\sum_{p \leq x} \frac{1}{p}\right)\right),$$

where we have used the bound $\log(1+x) \leq x$ for $x > -1$. Finally, using Mertens' second theorem (see, for instance, Theorem 427, [22]),

$$\sum_{p \le x} \frac{1}{p} = \log\log x + O(1)$$

for $x \ge 2$, we conclude that

$$\sum_{n \le x} h'(n) \le (\log x)^{c_r} = o(x)$$

as $x \to \infty$ where c_r is a constant depending only on r.

We now show that

$$\sum_{n=1}^{\infty} \frac{h'(n)}{n}$$

converges. We write

$$\sum_{n \le x} \frac{h'(n)}{n} \le \prod_{p \le x} \left(1 + \frac{h'(p)}{p} + \frac{h'(p^2)}{p^2} + \cdots + \frac{h'(p^{e_p})}{p^{e_p}}\right) \le \sum_{n \le x^{\pi(x)}} \frac{h'(n)}{n},$$

which on taking limits gives us

$$\sum_{n=1}^{\infty} \frac{h'(n)}{n} = \prod_{p} \left(1 + \sum_{i=1}^{\infty} \frac{h'(p^i)}{p^i}\right).$$

We next check that the product converges. We use estimate (2.5) so that

$$\frac{h'(p^i)}{p^i} = \frac{h_y^r(p^i) - h_y^r(p^{i-1})}{p^i} = O_r\left(\frac{1}{p^{i+1}}\right),$$

and so

$$\sum_{n=1}^{\infty} \frac{h'(n)}{n} = \lim_{x\to\infty} \prod \left(1 + O_r\left(\frac{1}{p^2}\right)\right) = \exp\left(O_r\left(\sum_p \frac{1}{p^2}\right)\right) = O_r(1).$$

We have proven the following result.

Proposition 2.15. *The moments of h_y exist and are given by the product over primes*

$$M_r(h_y) = \prod_p \left(1 + \sum_{i=1}^{\infty} \frac{h_y^r(p^i) - h_y^r(p^{i-1})}{p^i}\right).$$

In view of the above results, we arrive at the following result of Behrend [4].

Proposition 2.16. *For each integer $r \geq 1$, and $\alpha > 1$, we have*

$$\mathbf{d}\,\mathscr{A}_{y,\alpha} \leq F(y)\frac{M_r(h_y) - 1}{\alpha^r - 1}.$$

Proof. Since h is multiplicative and $h(p^e) > 1$ for all prime powers p^e, $h(n) \geq 1$ for all n. Then by the definition of h_y, $h_y(n) \geq 1$. Now we use Proposition 2.14 taking the function h_y^r for f and $\alpha_0 = 1$. This gives us an upper bound for the density of the set

$$\mathscr{H}_y = \{m : h_y(m) \geq 1\}.$$

It remains to show that $\mathbf{d}\,\mathscr{H}_y = F(y)\,\mathbf{d}\,\mathscr{A}_{y,\alpha}$. Since

$$\mathscr{H}_y = \bigsqcup_{P(n)\leq y} n\mathscr{A}_{y,\alpha},$$

we parrot the proof of Proposition 2.11. We have

$$\overline{\mathbf{d}}\left(\bigsqcup_{\substack{n>z\\P(n)\le y}} n\mathscr{A}_{y,\alpha}\right) \le \sum_{\substack{n>z\\P(n)\le y}} \overline{\mathbf{d}} n\mathscr{A}_{y,\alpha} \le \sum_{\substack{n>z\\P(n)\le y}} \frac{1}{n},$$

and since the limit of the right side is 0 as $z \to \infty$, this gives us

$$\mathbf{d}\,\mathscr{H}_y = \sum_{P(n)\le y} \mathbf{d}\, n\mathscr{A}_{y,\alpha} = \sum_{P(n)\le y} \frac{1}{n}\,\mathbf{d}\,\mathscr{A}_{y,\alpha} = F(y)\,\mathbf{d}\,\mathscr{A}_{y,\alpha}.$$

This establishes our result. □

We will refer to this result as the *Behrend moment method*. In the following chapter we will use the Behrend moment method to determine an upper bound for $\mathbf{d}\,\mathscr{A}$.

Chapter 3

The Deléglise program

The tightest bounds on the density of abundant numbers to date are due to Deléglise in 1998. He shows in [8] that

$$0.2474 < \mathbf{d}\mathscr{A} < 0.2480,$$

so that $\mathbf{d}\mathscr{A} = 0.247\ldots$, which is an improvement of two digits over the previous record. As was indicated in the previous chapter, the ideas used in the Deléglise program were based on the upper bound method used by Behrend. Given such a program, some natural questions arise. First, can the program compute arbitrarily many digits of the density given enough time? Assuming that the answer to this question is in the affirmative, it would be expected that with the improved computational speed of modern computers we would continue to see improvements on the bounds. However, the work up to Salié's in 1955 was all done by hand, then by Wall's computer in 1972, then finally by Deléglise's computer in 1998. Comparing this chronology to the size of the corresponding improvements, we get the sense that progress has not come rapidly. Could it be that existing techniques are inherently slow? In this chapter, we will investigate the answer to these two questions by studying the asymptotic complexity of the Deléglise program.

3.1 The Behrend-Deléglise method

Deléglise uses as his starting point the infinite sum expression for the density of $\mathscr{A}_\alpha$ (2.4),

$$\mathbf{d}\,\mathscr{A}_\alpha = \sum_{P(n)\le y} \frac{\mathbf{d}\,\mathscr{A}_{y,\alpha/h(n)}}{n},$$

where the sum is over all y-smooth numbers n. For a lower bound, two approximations are made. First, since each term of the sum is positive, a lower bound may be found by truncating the sum. Thus

$$\mathbf{d}\,\mathscr{A}_\alpha \ge \sum_{\substack{n\le z\\ P(n)\le y}} \frac{\mathbf{d}\,\mathscr{A}_{y,\alpha/h(n)}}{n}$$

for any choice of z. Second, we observe that whenever n is α-nondeficient, we have a simple expression for $\mathscr{A}_{y,\alpha/h(n)}$. Namely, since $\alpha/h(n) \le 1$, any number m has $h(m) \ge \alpha/h(n)$. Thus the members of the set $\mathscr{A}_{y,\alpha/h(n)}$ consists of all numbers m, $(m, \Pi(y)) = 1$. Since this set is periodic, we know by Lemma 1.3 that it has density $\varphi(\Pi(y))/\Pi(y) = F(y)$. This allows us to write

$$\mathbf{d}\,\mathscr{A}_\alpha \ge F(y) \sum_{\substack{n\le z\\ P(n)\le y\\ h(n)\ge\alpha}} \frac{1}{n}. \tag{3.1}$$

Note that this lower bound expression can be computed since it consists of a finite product and a finite sum. We will call this method of determining the lower bound for $\mathbf{d}\,\mathscr{A}_\alpha$ the *Behrend-Deléglise lower bound method*.

For an upper bound, Deléglise again uses the parameter z to split the infinite sum into two parts, according to whether $n \leq z$ or not. Thus

$$\mathbf{d}\,\mathscr{A}_\alpha = \sum_{\substack{n \leq z \\ P(n) \leq y}} \frac{\mathbf{d}\,\mathscr{A}_{y,\alpha/h(n)}}{n} + \sum_{\substack{n > z \\ P(n) \leq y}} \frac{\mathbf{d}\,\mathscr{A}_{y,\alpha/h(n)}}{n}, \tag{3.2}$$

with n being y-smooth. Since the first sum is finite, we may simply bound each of the terms of the sum from above. This is done by using the Behrend moment method of Proposition 2.16.

Let $\tilde{A}_{y,\alpha}$ be the minimum value among the function $F(y)$ and the numbers

$$F(y)\frac{M_r(h_y) - 1}{\alpha^r - 1}$$

for $r = 2^i$, $i = 0, 1, 2, \ldots, 12$. We have $\mathbf{d}\,\mathscr{A}_{y,\alpha} \leq \tilde{A}_{y,\alpha}$ and we use this estimate for the first sum in (3.2).

To bound the second sum in (3.2), we first bound $\mathbf{d}\,\mathscr{A}_{y,\alpha/h(n)}$ above by $F(y)$. Then together with the identity

$$F(y)^{-1} = \sum_{P(n) \leq y} \frac{1}{n}, \tag{3.3}$$

we have

$$\sum_{\substack{n > z \\ P(n) \leq y}} \frac{\mathbf{d}\,\mathscr{A}_{y,\alpha/h(n)}}{n} \leq F(y) \sum_{\substack{n > z \\ P(n) \leq y}} \frac{1}{n} = F(y)\left(F(y)^{-1} - \sum_{\substack{n \leq z \\ P(n) \leq y}} \frac{1}{n} \right), \tag{3.4}$$

giving us a bound for the infinite sum that is in terms of a finite sum. Combining the two estimates, we have the upper bound

$$\mathbf{d}\,\mathscr{A}_\alpha \leq \sum_{\substack{n\leq z\\ P(n)\leq y}} \frac{\tilde{A}_{y,\alpha/h(n)}}{n} + 1 - F(y) \sum_{\substack{n\leq z\\ P(n)\leq y}} \frac{1}{n}. \tag{3.5}$$

Thus both the upper and lower bounds are reduced to finite calculations that can be implemented on a computer. Analogously to the lower bound, we will call this method of determining an upper bound for $\mathbf{d}\,\mathscr{A}_\alpha$ the Behrend-Deléglise upper bound method.

3.2 The Deléglise program

We now describe how Deléglise implemented the foregoing ideas into a program. The prime array `prime` for primes in $[2, y]$ and the moments array `Lambda` for 2^ith moments of h_y for $i = 0, \ldots, 12$ are calculated first. In order to keep track of the y-smooth numbers n, the value of the current y-smooth number n being considered is stored in a variable `n` and the array `a` holds the exponents of the primes of n, so that we have access to both the value and factorization of n. We also need to keep track of the value of $\sigma(n)$. To do this, in the array `Pk` we hold the values of the prime powers for each prime dividing n, the array `Sk` holds the values $\sigma(p_k^{\texttt{a[k]}})$ for each prime $p_k \leq y$, and the array `sigma` holds the product of the first k entries in the array of `Sk`, so that $\sigma(n)$ can be found in the index of the prime $P(n)$. The function `init` initializes the first entry of each of these arrays to represent values corresponding to $n = 1$. Then a backtracking function `back` is used to run through all of the y-smooth numbers $\leq z$. Deléglise's function `back` is fairly intricate, we will reproduce his original C++ code below and give a small example of how it works. Here N=$\lfloor z \rfloor$ and K=$\pi(y)$.

```
void back(int k, Long n) {
  Long nextn;
  nextn = n;
  while (nextn <= N) {
    if(a[k]) {
      traite(k,nextn); // For computing bounds for density
    }
    if ((k < K) and (nextn*prime[k+1] <= N)) // Take care of overflow
      {
    a[k+1]=0;
    Pk[k+1] = 1;
    Sk[k+1] = 1;
    sigma[k+1] = sigma[k];
    back(k+1,nextn);
      }
    a[k]++;
    nextn = nextn * prime[k];
    Pk[k] *= prime[k];
  }
}
```

Suppose N $= 10$ and K $= 2$, so that we are looking for the 3-smooth numbers $n \leq 10$. We call `back(1,1)`. First, `nextn` is assigned the value 1. Next we enter the `while` loop since `nextn` $= 1 \leq 10$. Currently, `a[1]` $= 0$, so we do not enter the first if block. However, since $k < 2$ and $nextn \cdot prime[2] = 3 \leq 10$, both conditions of the second if statement are satisfied so we do enter the second if block. At this point the second entries of some arrays are initialized, and then back(2,1) is called.

The process repeats, but this time we do not enter the second if block since k is not less than 2. Instead, we move past this block and increment a[2] so that $a[2] = 1$. nextn is set to $1 \cdot 3$, and Pk[2] is set to 3. This brings us to the end of the while loop.

Since $nextn = 3 < 10$, we reenter the while block. This time $a[2] \neq 0$, so we enter the first if block. At this point we have found our first y-smooth number, stored as $nextn = 3$. The function traite, which is described in more detail below, computes the terms of the sums needed to determine the upper and lower bounds for the density of $\mathscr{A}_\alpha$. As before, we skip both if blocks. Then nextn is set to 9, and we return to the beginning of the while block.

We again enter the first if block and treat 9 as another y-smooth number. We also skip both if blocks and nextn is set to 27. However, this time when we check the while condition, it fails, so we exit the function call for back(2,1). At this point we return to the case $k = 1$ and $nextn = 1$. We increment a[1] to 1 and set $nextn = 2$. We return to the beginning of the while loop. Now 2 is counted among the y-smooths in the first if block and is treated accordingly. We also enter the second if block, where we call back(2,2).

Continuing to work through this process, we next find 6 and return from back(2,2). At the end of the while block we have $k = 1$ and $nextn = 1$. A couple more spins around the while block let us find 4 and 8. Then we return from back(1,1). Note that the number 1 is not included and so must be treated separately. Thus we have found all 3-smooth numbers greater than 1 and not greater than 10.

For a larger example, we provide the sequence of 10-smooth numbers from 2 up to 100 found by this algorithm. These are:

7, 49, 5, 35, 25, 3, 21, 15, 75, 9, 63, 45, 27, 81, 2, 14, 98, 10, 70, 50, 6, 42, 30, 18, 90, 54, 4, 28, 20, 100, 12, 84, 60, 36, 8, 56, 40, 24, 72, 16, 80, 48, 32, 96, 64.

We now describe the function `traite`, which is called whenever a y-smooth $n \leq z$ is found. This is the only part of Deléglise's code that distinguishes between calculating bounds for the density of α-abundant numbers versus abundant numbers. First, it tests n for α-abundancy. If n is α-nondeficient, the value $1/n$ is added to the running total `abundsum`. Otherwise, the value $1/n$ is added to the running total `defsum`. Next, the function `Ak` is called. This function computes the 13 upper bounds for $\mathbf{d}\,\mathscr{A}_{y,\alpha/h(n)}$ from each of the 13 moments in `Lambda` using the moment upper bound method of Proposition 2.16. Only the smallest value $\tilde{A}_{y,\alpha/h(n)}$ among these upper bounds is kept and returned. From the α-deficient value n, the value $\tilde{A}_{y,\alpha/h(n)}/n$ is calculated and added to the running total `Adefsum`.

After all of the y-smooth $n \leq z$ have been found, the program terminates after displaying the value of $F(y)\cdot$`abundsum` for the lower bound of $\mathbf{d}\,\mathscr{A}$, and the value of

$$\texttt{Adefsum} + 1 - F(y)(\texttt{defsum} + \texttt{abundsum}),$$

which is the upper bound for $\mathbf{d}\,\mathscr{A}_\alpha$.

3.3 Asymptotic complexity

We are now in a position to investigate the asymptotic complexity of the Deléglise program. We will also show that by an appropriate choice of parameters, the difference between the upper and lower bounds has limit 0 as the parameters are increased. Note that two parameters are involved, namely y, which bounds the smoothness of the y-smooth numbers n, and z, which bounds the size of n. In fact, there is a third parameter α, but this value does not affect the complexity of the program. Thus we will determine the running time of the program as well as the difference between upper and lower bounds in terms of y and z. This will allow us to choose optimal values of y and z and determine how such a choice will affect the running time for the program to calculate the value of $\mathbf{d}\,\mathscr{A}_\alpha$ to some given precision.

3.3.1 Running time

We first determine the running time $T(z, y)$ of the Deléglise program. The time complexity of the program is estimated by counting the computational steps which contribute to the running time, where the steps are defined to be arithmetical operations such as addition, multiplication, and division, comparisons, variable assignments, and function calls. Certain of these operations, namely addition, mutliplication, and division, are dependent on the number of bits of the numbers that are being operated upon. In particular, as described in Section 1.1 of [25], we have on k bit numbers that addition is $O(k)$, while division and multiplication are $O(k^2)$. For numbers near z, $k \asymp \log z$, so the more time consuming operations of multiplication and division take $O((\log z)^2)$ steps.

Program initialization consists of loading the precomputed arrays into memory as well as executing the init function. Since the arrays all have length $\pi(y)$, and the operations in the init function do not depend on z or y, initialization takes $O(y/\log y)$ steps. After the program initializes, the function back runs through the y-smooth numbers $n \leq z$. Comparing the cases of when n is abundant and n is deficient, we see that in either case there is a floating point calculation of $1/n$ and an addition to the running total of each of the respective cases. For the case when n is deficient there is an additional call to the function Ak, so this is the more time consuming case. However, this function takes time $O(1)$, as it does not depend on the values of z and y.

Thus the time complexity of the program is

$$T(z,y) = O((\log z)^2 \Psi(z,y)),$$

where the function $\Psi(z,y)$ is defined to be the number of $n \leq z$ that are y-smooth.

3.3.2 The error estimate

We let the *error* in the Deléglise method be the difference between the Deléglise upper and lower bounds, and denote the error $E(z,y)$. By taking the difference between (3.5) and (3.1), we determine the error to be

$$\begin{aligned} E(z,y) &= \sum_{\substack{n\leq z \\ P(n)\leq y}} \frac{\tilde{A}_{y,\alpha/h(n)}}{n} + 1 - F(y) \sum_{\substack{n\leq z \\ P(n)\leq y}} \frac{1}{n} - F(y) \sum_{\substack{n\leq z \\ P(n)\leq y \\ h(n)\geq\alpha}} \frac{1}{n} \\ &= \sum_{\substack{n\leq z \\ P(n)\leq y}} \frac{\tilde{A}_{y,\alpha/h(n)}}{n} + F(y) \sum_{\substack{n> z \\ P(n)\leq y}} \frac{1}{n} - F(y) \sum_{\substack{n\leq z \\ P(n)\leq y \\ h(n)\geq\alpha}} \frac{1}{n} \end{aligned}$$

$$= F(y) \sum_{\substack{n>z \\ P(n)\le y}} \frac{1}{n} + \sum_{\substack{n\le z \\ P(n)\le y \\ h(n)<\alpha}} \frac{\tilde{A}_{y,\alpha/h(n)}}{n}. \tag{3.6}$$

We will denote the two terms in the error

$$E_1(z,y) := F(y) \sum_{\substack{n>z \\ P(n)\le y}} \frac{1}{n} \tag{3.7}$$

and

$$E_2(z,y) := \sum_{\substack{n\le z \\ P(n)\le y \\ h(n)<\alpha}} \frac{\tilde{A}_{y,\alpha/h(n)}}{n}. \tag{3.8}$$

It is easy to see that (3.7) goes to zero as $z \to \infty$ since the infinite sum

$$\sum_{P(n)\le y} \frac{1}{n}$$

converges to $F(y)^{-1}$. However, we are interested in the rate at which this happens, so we will need results on the number of y-smooth numbers $n \le z$.

3.3.3 The estimate for E_1

The estimate for the sum

$$E_1(z,y) = F(y) \sum_{\substack{n>z \\ P(n)\le y}} \frac{1}{n}$$

in the Deléglise error will depend on the relative sizes of y and z. If y is less than $\exp((\log\log z)^2)$, we proceed as follows. We will split the sum into two sums, according to whether or not n is squarefree. We first estimate the sum

$$S(z,y) := \sum_{\substack{n>z \\ P(n)\le y}} \frac{|\mu(n)|}{n}.$$

Writing $u = (\log z)/\log y$, we have that $u > (\log z)/(\log\log z)^2$. We first show that

$$\sum_{\substack{n>z \\ P(n)\le y}} \frac{|\mu(n)|}{n} \le \sum_{j>u} \frac{1}{j!}\left(\sum_{p\le y}\frac{1}{p}\right)^j.$$

This is because, by the multinomial theorem, the expression

$$\left(\sum_{p\le y}\frac{1}{p}\right)^j$$

contains terms $1/m$ where m is y-smooth, squarefree, and $\omega(m) = j$ with multiplicity $j!$. Here the function $\omega(n)$ is the number of distinct prime divisors of n. The bound on the index $j > u$ excludes the numbers m with $\omega(m) \le u$ since if $j \le u$, then $y^j \le z$ so $m \le z$.

Next we use Mertens' theorem [22, Theorems 427, 428] which states that

$$\sum_{p\le x}\frac{1}{p} = \log\log x + B + o(1),$$

where B is the constant

$$B = \gamma + \sum_p \left(\log\left(1-\frac{1}{p}\right) + \frac{1}{p}\right)$$

and γ is the Euler-Mascheroni constant. From this, we deduce that we have the inequality

$$\sum_{p \le x} \frac{1}{p} \le c \log\log x$$

for some absolute constant $c > 0$. For instance, from [29] we deduce that we can take $c = 19.5$, valid for all $x > 1$, or using [10] we see that we can take $c = 1.0999$ for $x \ge 10^6$.

We may now bound

$$\sum_{j>u} \frac{1}{j!} \left(\sum_{p \le y} \frac{1}{p} \right)^j \le \sum_{j>u} \frac{1}{j!} \left(c \log\log y \right)^j . \tag{3.9}$$

Note that in our range where $y < \exp((\log\log z)^2)$, that is to say $\log y < (\log\log z)^2$, we have

$$\log y \log\log y < 2(\log\log z)^2 \log\log\log z,$$

where the right side of the inequality is asymptotically smaller than $\log z$. Since $u = \log z / \log y$, we conclude that $2c \log\log y \le u$. Now for any u bounded below by $2c \log\log y$, we have that the ratio of consecutive terms of the sum on the right side of (3.9) is

$$\frac{\frac{1}{(j+1)!} \left(c \log\log y \right)^{j+1}}{\frac{1}{j!} \left(c \log\log y \right)^j} = \frac{c \log\log y}{j+1} \le \frac{c \log\log y}{2c \log\log y} = \frac{1}{2}.$$

Thus we have

$$\sum_{j>u} \frac{1}{j!} \left(c \log\log y \right)^j \le \sum_{i \ge 0} \frac{1}{\lceil u \rceil !} (c \log\log y)^{\lceil u \rceil} \frac{1}{2^i} = 2 \frac{1}{\lceil u \rceil !} (c \log\log y)^{\lceil u \rceil}.$$

To bound the factorial, we observe that for $n \geq 1$,

$$\log n! \geq \int_1^n \log t \, dt = n \log n - n + 1,$$

so

$$n! > e \left(\frac{n}{e} \right)^n . \tag{3.10}$$

Thus

$$S(z, y) \leq \left(\frac{e}{u} \right)^u (c \log \log y)^{u+1}.$$

Since our bounds on y and u imply $\log \log y \leq 2 \log \log \log z$ and $\log \log \log z \ll \log \log u$, we may write

$$\sum_{\substack{n > z \\ P(n) \leq y}} \frac{|\mu(n)|}{n} \ll \log \log u \left(\frac{c' \log \log u}{u} \right)^u \tag{3.11}$$

for some constant c' when $y < \exp((\log \log z)^2)$.

Now we estimate the remaining terms, namely the terms corresponding to n not squarefree. We use the observation that a number n can be decomposed uniquely into a square part and a squarefree part, as follows. Let m^2 be the largest square divisor of n. Then writing $v = n/m^2$, $n = m^2 v$, we have v is squarefree, for if not, then a square divisor $d^2 > 1$ of v can be found and $(md)^2$ is a square divisor of n larger than m^2, a contradiction. Then we estimate separately the sums over n according to whether $m > n^{1/4}$ or $m \leq n^{1/4}$. In the former case,

$$\sum_{\substack{n > z \\ P(n) \leq y \\ m > n^{1/4}}} \frac{1}{n} \leq \sum_{m > z^{1/4}} \frac{1}{m^2} \sum_{P(v) \leq y} \frac{1}{v}$$

$$\ll \frac{1}{z^{1/4}} \cdot \log y. \tag{3.12}$$

When $m \leq n^{1/4}$,

$$\sum_{\substack{n>z \\ P(n)\leq y \\ m\leq n^{1/4}}} \frac{1}{n} \leq \sum_{m} \frac{1}{m^2} \sum_{\substack{P(v)\leq y \\ v>z^{1/2}}} \frac{1}{v}$$
$$\ll \sum_{\substack{P(n)\leq y \\ n>z^{1/2}}} \frac{|\mu(n)|}{n}.$$

Thus this sum is of greater order than (3.11), and since $\frac{\log z^{1/2}}{\log y} = u/2$, we may use (3.11) to estimate

$$\sum_{\substack{P(n)\leq y \\ n>z^{1/2}}} \frac{|\mu(n)|}{n} \ll \log\log(u/2) \left(\frac{c' \log\log(u/2)}{u/2}\right)^{u/2} \ll \frac{1}{u^{u/2}}.$$

We conclude that for $y < \exp((\log\log z)^2)$,

$$\sum_{\substack{n>z \\ P(n)\leq y}} \frac{1}{n} \ll \frac{1}{u^{u/2}} + \frac{\log y}{z^{1/4}}, \tag{3.13}$$

and so

$$F(y) \sum_{\substack{n>z \\ P(n)\leq y}} \frac{1}{n} \ll \frac{1}{\log y} \cdot \frac{1}{u^{u/2}} + \frac{1}{z^{1/4}}$$

when $y < \exp((\log\log z)^2)$.

When $y \geq \exp((\log\log z)^2)$, namely when $u \leq (\log z)/(\log\log z)^2$, we will use published results on the behavior of the function $\Psi(z, y)$, which counts the y-smooth numbers $n \leq z$. This function has been studied extensively; see, for instance, [26, 34, 21]. In particular, we will need an upper bound for $\Psi(z, y)$. Since $u = (\log z)/\log y$, the Ψ function can be written as $\Psi(z, z^{1/u})$. The work of de Bruijn related the Ψ function for certain y and z to the Dickman ρ-function, which is defined for $u \geq 0$ as the unique continuous solution to the differential-difference equation

$$u\rho'(u) = -\rho(u-1), \qquad u > 1$$

satisfying the initial condition

$$\rho(u) = 1, \qquad 0 \leq u \leq 1.$$

It was proven by de Bruijn [7] that

$$\rho(u) = \exp\left\{-u\left(\log u + \log\log u - 1 - \frac{1}{\log u} + \frac{\log\log u}{\log u} + O\left(\frac{(\log\log u)^2}{(\log u)^2}\right)\right)\right\}$$

as $u \to \infty$. Later Hildebrand [23] was able to prove the following.

Theorem 3.1 (Hildebrand)**.** *Let $\epsilon > 0$. Uniformly under the condition $z \geq 2$, $1 \leq u \leq (\log z)/(\log\log z)^{5/3+\epsilon}$, we have*

$$\Psi(z, z^{1/u}) = z\rho(u)\left(1 + O_\epsilon\left(\frac{u\log(u+1)}{\log z}\right)\right).$$

Combining the two results, we have for $z \geq 2$, $1 \leq u \leq (\log z)/(\log\log z)^{5/3+\epsilon}$, that

$$\Psi(z, y) \ll zu^{-u}.$$

Now we are prepared to estimate the sum

$$\sum_{\substack{n>z \\ P(n)\le y}} \frac{1}{n} \tag{3.14}$$

for the case $y \ge \exp((\log\log z)^2)$. The strategy will be to split the sum according to whether n is larger or smaller than the bound $z_0 = (\exp\exp\sqrt{\log y})^2$, since for y-smooth $n \le z_0$ the Hildebrand estimate applies, while for y-smooth $n > z_0$ our earlier estimate (3.13) applies since $y = \exp((\log\log\sqrt{z_0})^2) < \exp((\log\log z_0)^2)$. Thus we separate the sum

$$\sum_{\substack{n>z \\ P(n)\le y}} \frac{1}{n} = \sum_{\substack{n\in(z,z_0] \\ P(n)\le y}} \frac{1}{n} + \sum_{\substack{n>z_0 \\ P(n)\le y}} \frac{1}{n}. \tag{3.15}$$

For the first sum above, we have

$$\begin{aligned}
\sum_{\substack{n\in(z,z_0] \\ P(n)\le y}} \frac{1}{n} &= \int_z^{z_0} \frac{d\Psi(t,y)}{t} \\
&= \left[\frac{\Psi(t,y)}{t}\right]_z^{z_0} + \int_z^{z_0} \frac{\Psi(t,y)}{t^2}dt \\
&= \frac{\Psi(z_0,y)}{z_0} - \frac{\Psi(z,y)}{z} + \int_z^{z_0} \frac{\Psi(t,y)}{t^2}dt \\
&\le \frac{\Psi(z_0,y)}{z_0} + \int_z^{z_0} \frac{\Psi(t,y)}{t^2}dt \\
&\ll \frac{1}{u_0^{u_0}} + \int_z^{\infty} \frac{\rho(v)}{t}dt \quad \left(\text{where } u_0 = \frac{\log z_0}{\log y} \text{ and } v = \frac{\log t}{\log y}\right), \\
&= \frac{1}{u_0^{u_0}} + \log y \int_u^{\infty} \rho(v)dv \\
&\ll \frac{\log y}{u^u}.
\end{aligned}$$

For the second sum in (3.15), we use the estimate (3.13) so that, again taking $u_0 =$

$(\log z_0)/\log y$, we have

$$\sum_{\substack{n>z_0\\P(n)\le y}} \frac{1}{n} \ll \frac{1}{{u_0}^{u_0/2}} + \frac{\log y}{z_0^{1/4}}. \tag{3.16}$$

Since

$$\frac{1}{{u_0}^{u_0/2}} < \frac{1}{(u_0/2)^{u_0/2}}$$

and

$$\frac{1}{u^u} < \frac{\log y}{u^u}$$

for $y > e$, we now wish to show that

$$\frac{1}{(u_0/2)^{u_0/2}} < \frac{1}{u^u},$$

or, equivalently, that $u < u_0/2$. This in turn is equivalent to

$$\log z < \frac{\log((\exp\exp\sqrt{\log y})^2)}{2} = \exp\sqrt{\log y} \tag{3.17}$$

and finally,

$$\log y > (\log\log z)^2, \tag{3.18}$$

which is the case we are considering. Thus, we conclude that

$$\frac{1}{u_0^{u_0/2}} \ll \frac{1}{u^u},$$

and that

$$\sum_{\substack{n>z\\P(n)\le y}} \frac{1}{n} \ll \frac{\log y}{u^u} + \frac{\log y}{z_0^{1/4}}.$$

We next show that

$$\frac{\log y}{z_0^{1/4}} < \frac{\log y}{u^u}.$$

This amounts to showing that

$$u^u < z_0^{1/4}.$$

We use (3.17) so that

$$u = \frac{\log z}{\log y} < \frac{\exp\sqrt{\log y}}{\log y},$$

which gives

$$u^u < \left(\frac{\exp\sqrt{\log y}}{\log y}\right)^{\frac{\exp\sqrt{\log y}}{\log y}}.$$

From the definition of z_0 we have

$$z_0^{1/4} = \sqrt{\exp\exp\sqrt{\log y}}.$$

Thus we wish to show that

$$\left(\frac{\exp\sqrt{\log y}}{\log y}\right)^{\frac{\exp\sqrt{\log y}}{\log y}} < \sqrt{\exp\exp\sqrt{\log y}}.$$

Taking logs, we need that

$$\frac{\exp\sqrt{\log y}}{\log y}(\sqrt{\log y} - \log\log y) < \frac{1}{2}\exp\sqrt{\log y}.$$

For this inequality to hold, it suffices to show that

$$\frac{\exp\sqrt{\log y}}{\sqrt{\log y}} = \frac{\exp\sqrt{\log y}}{\log y}\sqrt{\log y} < \frac{1}{2}\exp\sqrt{\log y}.$$

It is clear that this inequality holds for $y > e^4 \approx 54.6$. We conclude that

$$F(y) \sum_{\substack{n>z \\ P(n)\leq y}} \frac{1}{n} \ll \frac{1}{u^u}$$

for $y \geq \exp((\log\log z)^2)$.

Now we compare the two bounds. In the case of the first bound where $\log y < (\log\log z)^2$ and $u > (\log z)/(\log\log z)^2$, we have

$$\frac{1}{\log y} > \frac{1}{(\log\log z)^2} > \left(\frac{(\log\log z)^2}{\log z}\right)^{\frac{\log z}{2(\log\log z)^2}} > \frac{1}{u^{u/2}}.$$

This gives us that

$$\frac{1}{\log y} \cdot \frac{1}{u^{u/2}} > \left(\frac{1}{u^{u/2}}\right)^2 = \frac{1}{u^u}.$$

Thus we see that for sufficiently large z, the second bound is preferable. Thus we will restrict our consideration to the latter case where $y \geq \exp((\log\log z)^2)$ and use the following bound.

Proposition 3.2. *Let* $y \geq \exp((\log\log z)^2)$. *Then*

$$E_1(z, y) = F(y) \sum_{\substack{n>z \\ P(n)\leq y}} \frac{1}{n} \ll \frac{1}{u^u},$$

where $u = (\log z)/\log y$.

3.3.4 The second sum

We now estimate the second sum in (3.6), namely (3.8), which we reproduce here:

$$E_2(z,y) = \sum_{\substack{n\le z\\ P(n)\le y\\ h(n)\le\alpha}} \frac{\tilde{A}_{y,\alpha/h(n)}}{n}.$$

We first estimate $\tilde{A}_{y,\alpha}$. Since $\tilde{A}_{y,\alpha}$ is defined as the minimum value among the members of a set of upper bound values, we have a simple upper bound for this function by taking the minimum value of the members of a subset of the original set of upper bounds. In particular, we choose the trivial upper bound $F(y)$ and the first moment bound. In order to determine the value of u at which we switch from one bound to the other we will use a parameter w, so that when $u < 1 + 1/w$ we use the trivial bound and when $u \ge 1 + 1/w$ we use the first moment bound.

We will also need to estimate $M_1(h_y)$. Recall that $M_1(h_y)$ has an Euler product,

$$M_1(h_y) = \prod_p \left(1 + \sum_{i=1}^{\infty} \frac{h_y(p^i) - h_y(p^{i-1})}{p^i}\right).$$

Since $h_y(p^i) - h_y(p^{i-1}) = 1/p^i$ when $p > y$ and is 0 when $p \le y$, we have

$$\begin{aligned} M_1(h_y) &= \prod_{p>y}\left(1 + \sum_{i=1}^{\infty}\frac{1}{p^{2i}}\right) \\ &= \prod_{p>y}\left(1+\frac{1}{p^2-1}\right) \\ &= \exp\sum_{p>y}\log\left(1+\frac{1}{p^2-1}\right). \end{aligned}$$

We upper bound the expression $\log(1+1/(p^2-1))$ using the bound $\log(1+x) \le x$

for $x > 0$. We wish to bound it below by

$$\frac{1}{p^2} < \log\left(1 + \frac{1}{p^2 - 1}\right) = \log\left(\frac{1}{1 - \frac{1}{p^2}}\right).$$

This can be seen by taking exponentials of both sides and comparing the Maclaurin expansions term-by-term:

$$\exp\left(\frac{1}{p^2}\right) = 1 + \frac{1}{p^2} + \frac{1}{2p^4} + \frac{1}{6p^6} + \cdots,$$

while

$$\frac{1}{1 - \frac{1}{p^2}} = 1 + \frac{1}{p^2} + \frac{1}{p^4} + \frac{1}{p^6} + \cdots.$$

Thus,

$$\exp\left(\sum_{p>y} \frac{1}{p^2}\right) \leq M_1(h_y) \leq \exp\left(\sum_{p>y} \frac{1}{p^2 - 1}\right),$$

and we conclude that

$$M_1(h_y) - 1 \sim \frac{1}{y \log y}$$

as $y \to \infty$. We now consider the sum over those n such that $\alpha/h(n) \in [1, 1 + 1/w]$. Here, we use the upper bound $\tilde{A}_{y,\alpha} \leq F(y)$. We will also make use of the following result of Erdős [15].

Theorem 3.3 (Erdős)**.** *Let $N(x; a, b)$ denote the number of $n \leq x$ such that $a \leq h(n) < b$. Then there is an absolute constant c such that for $a \geq 1$ and $x > t$,*

$$N\left(x; a, a + \frac{1}{t}\right) < c\frac{x}{\log t}.$$

We now use the above theorem and partial summation to bound the sum

$$\sum_{\substack{n\le z\\ 1\le\alpha/h(n)<1+1/w}} \frac{1}{n}.$$

Writing the inequality $1 \le \alpha/h(n) < 1 + 1/w$ as

$$\alpha - \frac{\alpha}{w+1} = \alpha\left(1+\frac{1}{w}\right)^{-1} < h(n) \le \alpha,$$

we have

$$\begin{aligned}\sum_{\substack{n\le z\\ 1\le\alpha/h(n)<1+1/w}} \frac{1}{n} &= 1 + \int_1^z \frac{dN(t;\alpha-\frac{\alpha}{w+1},\alpha)}{t}\\ &= 1 + \frac{N(t;\alpha-\frac{\alpha}{w+1},\alpha)}{t}\bigg|_1^z + \int_1^z \frac{N(t;\alpha-\frac{\alpha}{w+1},\alpha)}{t^2}dt\\ &= 1 + O\left(\frac{1}{\log w}\right) + \int_1^z \frac{N(t;\alpha-\frac{\alpha}{w+1},\alpha)}{t^2}dt\end{aligned}$$

for $w \ge 2$. The integral can be estimated by noting that

$$\int_1^z \frac{N(t;\alpha-\frac{\alpha}{w+1},\alpha)}{t^2}dt = \int_1^z O\left(\frac{t}{t^2\log w}\right)dt = O\left(\frac{1}{\log w}\int_1^z \frac{1}{t}dt\right) = O\left(\frac{\log z}{\log w}\right)$$

for $w, z \ge 2$. Thus, we have

$$F(y)\sum_{\substack{n\le z\\ P(n)\le y\\ 1\le\alpha/h(n)<1+1/w}} \frac{1}{n} = O\left(\frac{u}{\log w}\right)$$

for $w, y, z \ge 2$.

For the remaining case where $\alpha/h(n) \geq 1 + 1/w$, we use the first moment upper bound, so that

$$\tilde{A}_{y,\alpha/h(n)} \leq F(y)\frac{M_1(h_y) - 1}{\frac{\alpha}{h(n)} - 1} = O\left(\frac{1}{\log y} \cdot \frac{1}{y \log y} \cdot w\right) = O\left(\frac{w}{y(\log y)^2}\right)$$

for $w, y \geq 2$. Since

$$\sum_{\substack{n \leq z \\ P(n) \leq y}} \frac{1}{n} \leq \frac{1}{F(y)},$$

we have that the sum

$$\sum_{\substack{n \leq z \\ P(n) \leq y \\ \alpha/h(n) \geq 1+1/w}} \frac{\tilde{A}_{y,\alpha/h(n)}}{n}$$

in this case is

$$O\left(\frac{w}{y(\log y)^2}\right) \cdot \sum_{\substack{n \leq z \\ P(n) \leq y}} \frac{1}{n} = O\left(\frac{w}{F(y)y(\log y)^2}\right) = O\left(\frac{w}{y \log y}\right)$$

for $w, y \geq 2$. Thus the second sum has the combined estimate of

$$O\left(\frac{u}{\log w} + \frac{w}{y \log y}\right)$$

for $u \geq 1$ and $w, y \geq 2$. Setting the two terms equal to each other, we see that we want to choose a value of w so that

$$y \log z \asymp w \log w.$$

If we choose $w = yu$, the right side is of order $yu \log y = y \log z$ if $y > u$. With this choice of w, we have that our sum is bounded by

$$\sum_{\substack{n\le z\\ P(n)\le y\\ h(n)\le\alpha}} \frac{\tilde{A}_{y,\alpha/h(n)}}{n} = O\left(\frac{u}{\log y}\right),$$

for $u \ge 1$, $y \ge 2$, and $y > u$. In fact, if we simply choose $w = y$ we arrive at the same estimate, so the condition $y > u$ is superfluous. We have proven the following proposition.

Proposition 3.4. *For $u \ge 1$ and $y \ge 2$,*

$$\sum_{\substack{n\le z\\ P(n)\le y\\ h(n)\le\alpha}} \frac{\tilde{A}_{y,\alpha/h(n)}}{n} = O\left(\frac{u}{\log y}\right).$$

3.3.5 Asymptotic complexity

We now combine the two error terms. For the first error term we have restricted ourselves to the case $y \ge \exp((\log\log z)^2)$ where the bound is

$$\frac{1}{u^u}.$$

Collecting the two error terms, we have the error bound

$$O\left(\frac{1}{u^u} + \frac{u}{\log y}\right).$$

We now seek a relationship between y and z which minimizes this upper bound. Equating the terms suggests we take $\log y = u^{u+1}$. Taking logs, we have

$$\log\log y = u\log u + \log u,$$

while taking logs a second time we have

$$\log\log\log y = \log u + O(\log\log u).$$

The quotient of the two expressions gives

$$\frac{\log\log y}{\log\log\log y} = u + O\left(\frac{u\log\log u}{\log u}\right).$$

Solving the expression

$$\frac{\log\log y}{\log\log\log y} = \frac{\log z}{\log y}$$

for z, we have

$$z = y^{\frac{\log\log y}{\log\log\log y}}.$$

Thus we have the following theorem.

Theorem 3.5. *With parameters y, z chosen to be $z = y^{\frac{\log\log y}{\log\log\log y}}$ so that*

$$u = \frac{\log\log y}{\log\log\log y},$$

we have

$$E(z, y) \ll \frac{1}{u^u}.$$

This establishes that given the appropriate choice of y and z, the Deléglise program can calculate the density $\mathbf{d}\,\mathscr{A}_\alpha$ to any precision.

The time complexity of the Deléglise program is $\ll (\log z)^2\Psi(z,y)$, which is asymptotic to $z(\log z)^2\rho(u)$ as $u \to \infty$ for our range of u. We can use this to determine a bound on the time it would take to estimate $\mathbf{d}\,\mathscr{A}_\alpha$ within $1/10^k$ for any given k. Thus taking t to be the running time of the program, and $z = y^{\frac{\log\log y}{\log\log\log y}}$ as in the theorem above, we have

$$\begin{aligned} t &\ll (\log z)^2\Psi(z,y) \\ &\ll \frac{z(\log z)^2}{u^u} \\ &= \left(\frac{\log y \log\log y}{\log\log\log y}\right)^2 \left(\frac{y\log\log\log y}{\log\log y}\right)^{\frac{\log\log y}{\log\log\log y}} \\ &= \exp\left(O\left(\frac{\log y \log\log y}{\log\log\log y}\right)\right). \end{aligned}$$

Next we write

$$\frac{1}{10^k} = O\left(\frac{1}{u^u}\right),$$

so that

$$u^u \ll 10^k,$$

and

$$u\log u = O(k).$$

Now since we have $u = (\log\log y)/\log\log\log y$, taking logs we have

$$\log u = \log\log\log y - \log\log\log\log y,$$

so

$$u \log u = \log\log y - \frac{\log\log y \log\log\log\log y}{\log\log\log y}.$$

Thus $u \log u \asymp \log\log y$, so

$$\log\log y = O(k).$$

Then

$$\log y = e^{O(k)}$$

and

$$t \leq \exp\exp(O(k)).$$

Thus we have proven the following corollary.

Corollary 3.6. *Let t be the time that the Deléglise algorithm takes to determine the density $\mathbf{d}\,\mathscr{A}_\alpha$ to within 10^{-k}. Then we have that*

$$t < e^{e^{ck}},$$

where c is an absolute constant.

Note that we have proven only an upper bound for t and not a lower bound. In order to determine a lower bound, we would need a lower bound estimate of the error $E(z, y)$, which is an area for further research. However, if the corollary reflects the true order of magnitude of the time complexity, the double exponential character of the time bound would explain the slow progress made in the estimation of $\mathbf{d}\,\mathscr{A} = \mathbf{d}\,\mathscr{A}_2$.

Chapter 4

Improvements to the Deléglise algorithm

As we have shown in the previous chapter, although the program used by Deléglise is guaranteed to determine the density of abundant numbers to within k decimal places for any k, the running time may increase double-exponentially in k. In this case it would quickly become prohibitive to determine successive digits for the density simply by increasing the values of the parameters y and z. In this chapter, we study the sources of error contributing to the size of the error function $G(z, y)$ and use various ideas to reduce the size of the error. In particular we will be able to determine the next decimal digit for $\mathbf{d}\mathscr{A}$.

4.1 Some lower bound improvements

Let $\mathscr{A}_D(w, z, y)$ denote the set of nondeficient numbers of the form uv, where u is a y-smooth number with $u \in (w, z]$, and v is a number relatively prime to $\Pi(y)$. The

Behrend-Deléglise lower bound calculates the density of the subset $\mathscr{A}_D(1, z, y)$. One idea for improving the lower bound is to add to $\mathbf{d}\,\mathscr{A}_D(1, z, y)$ the density of other subsets of $\mathscr{A}'$ that are pairwise disjoint. We will describe several such subsets and discuss how to calculate their densities.

4.1.1 The small primes method

Consider the set $2\mathscr{A}_D(1, z, y)$ for $2 \leq y$. Each member is clearly nondeficient, in fact, abundant. If we know the density of $\mathscr{A}_D(1, z, y)$, it is easy to find the density of $2\mathscr{A}_D(1, z, y)$ since we can simply multiply the former by $1/2$. The only problem is that we require the set to be disjoint from $\mathscr{A}_D(1, z, y)$. This motivates the consideration of the set $2\mathscr{A}_D(1, z, y) \setminus \mathscr{A}_D(1, z, y) = 2\mathscr{A}_D(z/2, z, y)$. Computationally, $\mathbf{d}\,\mathscr{A}_D(z/2, z, y)$ can be found easily since this value is a subsum of the sum that calculates $\mathbf{d}\,\mathscr{A}_D(1, z, y)$. Thus, it is only necessary for a program to keep track of two sums instead of one, and test term-by-term whether or not a y-smooth number is less than $z/2$, and add values to the two sums accordingly.

This idea extends to higher powers of 2. Indeed, the set $4\mathscr{A}_D(z/2, z, y)$ is disjoint with each of $\mathscr{A}_D(z, y)$ and $2\mathscr{A}_D(z/2, z, y)$. Moreover, note that finding the density of $4\mathscr{A}_D(z/2, z, y)$ requires no extra computations aside from a division by 4 once $\mathbf{d}\,\mathscr{A}_D(z/2, z, y)$ has been found. Continuing in this manner, we have a sequence of sets $S_i^2 = 2^i\mathscr{A}_D(z/2, z, y)$ that are pairwise disjoint having known densities. Since densities are not necessarily infinitely additive, it remains to check that

$$\sum_{i=0}^{\infty} \mathbf{d}\, S_i^2 = \mathbf{d}\, S_0^2 \sum_{i=0}^{\infty} \frac{1}{2^i} = 2\,\mathbf{d}\, S_0^2.$$

Thus, we must show that

$$\overline{\mathbf{d}} \bigcup_{i>k} S_i^2 \to 0$$

as $k \to \infty$. This can be seen by noting that $S_i \subseteq 2^i\mathbb{N}$, so

$$\bigcup_{i>k} S_i \subseteq \bigcup_{i>k} 2^i\mathbb{N} = 2^{k+1}\mathbb{N}.$$

Then

$$\overline{\mathbf{d}} \left(\bigcup_{i>k} S_i^2 \right) \leq \mathbf{d}\, 2^{k+1}\mathbb{N} = \frac{1}{2^{k+1}}.$$

This can also be seen by observing that if sets S_i^2 have densities and are disjoint, then the lower densities of their union is at least the sum of their densities. Since the limit of the sequence on the right as $k \to \infty$ is 0, this establishes the infinite density sum expression. Note that the net increase of the new lower bound above the original lower bound is $\mathbf{d}\, S_0^2$.

An analogous argument involving odd members of $\mathscr{A}_D(1, z, y)$ and powers of 3 can also be used, giving a smaller but still noticeable improvement in the lower bound. In this case, we begin with the odd y-smooth nondeficient numbers $n \in (z/3, z]$ and define

$$S_i^3 = 3^i(\mathscr{A}_d(z/3, z, y) \setminus 2\mathbb{N}).$$

Repeating the argument yields

$$\sum_{i=0}^{\infty} \mathbf{d}\, S_i^3 = \mathbf{d}\, S_0^3 \sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{3}{2}\mathbf{d}\, S_0^3.$$

In general, for a prime $p \le y$, we construct sets

$$S_i^p = p^i \left(\mathscr{A}_D^p(z/p, z, y) \setminus \bigcup_{q<p} q\mathbb{N} \right)$$

with density sum having value

$$\mathbf{d}\, S_0^p \cdot \frac{p}{p-1}.$$

Then for any bound $y_0 \le y$, we may sum these densities over the primes $p \le y_0$ to get

$$\sum_{p \le y_0} \mathbf{d}\, S_0^p \cdot \frac{p}{p-1}.$$

We will call this method of using small primes p to augment the value of the original density the *small primes method*.

Using the small primes method for the single prime $p = 2$, we find for $y = 500$ and $z = 10^{14}$ that

$$\mathbf{d}\,\mathscr{A} \ge 0.247460540\ldots,$$

which is an improvement of about 9.16×10^{-6} over the Deléglise lower bound of $0.247451383\ldots$. If we also use the small primes method for the next prime $p = 3$, we improve this to

$$\mathbf{d}\,\mathscr{A} \ge 0.247461012\ldots,$$

which improves on the small primes result for $p = 2$ by about 4.72×10^{-7}.

4.1.2 The medium primes method

For a subset S of natural numbers, we introduce the notation $\mathscr{M}_y(S)$ to denote the set of all multiples ms of each $s \in S$, where m is a natural number such that

$(m, \Pi(y)) = 1$. Recall from Section 2.1 the notation $p(n)$ and $P(n)$ for the smallest and largest prime factors of n, respectively. Consider a y-smooth number n not greater than z and a number m such that $p(m) \geq y_n$, where

$$y_n = \begin{cases} \max\{P(n), (z+1)/n\} & \text{if } n > z/y, \\ y & \text{if } n \leq z/y. \end{cases}$$

Note that the numbers mn as defined are exactly the members of the set $\mathscr{M}_{y_n}(\{n\})$, which has density

$$\mathbf{d}\,\mathscr{M}_{y_n}(\{n\}) = \frac{F(y_n)}{n}.$$

Lemma 4.1. *The sets*

$$\mathscr{M}_{y_n}(\{n\})$$

over all y-smooth numbers n not greater than z are disjoint.

Proof. We will show that, given a number mn such that n is y-smooth and not greater than z, and m has $p(m) \geq y_n$, we can retrieve the number n, so that mn belongs only to $\mathscr{M}_{y_n}(\{n\})$. Write $mn = p_1 p_2 p_3 \ldots p_k$ as a product of primes such that $p_i \leq p_{i+1}$ for $i = 1, 2, \ldots, k-1$, and $n_j = p_1 p_2 \ldots p_j$. Thus n must be one of n_j, $j = 0, 1, \ldots, k$.

Let n_j be the y-smooth part of mn. First suppose that $n_j > z$. Then since $n \leq z$ it must be that $n \neq n_j$ so $y_n < y$ and $n > z/y$. Let i be such that $n_i \leq z < n_i p_{i+1}$. Note that either $n = n_i$ or $np(m) \leq n_i$. But $np(m) \geq z + 1 > n_i$, so $n = n_i$. Next we suppose $n_j \leq z$. We will show that n cannot be greater than z/y. Suppose to the contrary. Then $y_n = \max\{P(n), (z+1)/n\}$, so $z + 1 \leq ny_n \leq np(m) \leq n_j \leq z$, a contradiction. Thus $n \leq z/y$ and so $n = n_j$. □

Now if n is nondeficient, then every member of $\mathscr{M}_{y_n}(\{n\})$ is nondeficient. We will denote by $T_0(z, y)$ the set

$$T_0(z, y) := \bigcup_{\substack{n \le z \\ P(n) \le y \\ h(n) \ge 2}} \mathscr{M}_{y_n}(\{n\}).$$

Note that $\mathscr{A}_D(z, y) \subseteq T_0(z, y)$, so that we can expect an improvement in our lower bound for the density of abundant numbers,

$$\mathbf{d}\,\mathscr{A} \ge \mathbf{d}\,T_0(z, y).$$

We will call this the *medium primes method*.

Using the medium primes method, we find for $z = 10^{14}$ and $y = 500$ that

$$\mathbf{d}\,\mathscr{A} \ge 0.24747574.$$

This is an improvement of about 2.43×10^{-5} over the Deléglise lower bound. Comparing the medium primes lower bound to the small primes lower bound, we see that the medium primes method is an improvement of about 1.47×10^{-5}. Thus for our choices of z and y, the medium primes method is preferable.

Unfortunately, the small primes and medium primes methods are incompatible. For instance, the number $n_1 m_1$ with $n_1 = 2^{44} \cdot 3$ and $m_1 = 5$ is considered in the medium primes method for $z = 10^{14}, y = 500$. However, $n_1 m_1 = 2^2 n_2$ where $n_2 = 2^{42} \cdot 3 \cdot 5$. This number is considered in the small primes method since $z/2 < n_2 \le z$. Thus the two methods do not consider disjoint sets of abundant numbers so we may not combine the two improvements.

4.1.3 The large primes method

Suppose n is y-smooth, $n \leq z$, and n is deficient. Then the nondeficient multiples nm, where $(m, \Pi(y)) = 1$, are not accounted for in the Deléglise lower bound density. We now capture the density of subsets of these nondeficient numbers which involve particulary simple calculations.

The single large primes method. We first consider y-smooth numbers n not greater than z, such that

$$2\left(1 - \frac{1}{y+1}\right) \leq h(n) < 2.$$

Then n is deficient, but if there is a prime p such that

$$y < p \leq \frac{h(n)}{2 - h(n)},$$

then np is nondeficient. We can see this since $(n, p) = 1$, so we have

$$h(np) = h(n)h(p) = h(n)\left(1 + \frac{1}{p}\right) \geq h(n)\left(1 + \frac{2 - h(n)}{h(n)}\right) = 2.$$

Since np is nondeficient, any multiple of np is also nondeficient. However, we wish to use these new numbers in conjunction with the members of $\mathscr{A}_D(z, y)$ from the original Behrend-Deléglise method. In fact, we will show that the numbers are not members of $T_0(z, y)$ considered in the medium primes method. We must also ensure that various choices of n and p do not conflict with each other. Thus we restrict our attention to multiples mnp where $(m, \Pi(p - 1)) = 1$, namely the sets

$\mathscr{M}_{p-1}(\{np\})$, where the notation $\mathscr{M}_y$ is defined in Subsection 4.1.2. We now establish the compatibility of all of these sets.

Lemma 4.2. *For each y-smooth n not greater than z such that*

$$2\left(1-\frac{1}{y+1}\right) \leq h(n) < 2,$$

and for each prime p such that

$$y < p \leq \frac{h(n)}{2-h(n)},$$

the sets

$$\mathscr{M}_{p-1}(\{np\})$$

are pairwise disjoint. In addition, each such set is disjoint from $T_0(z, y)$.

Proof. We first show that if we fix an appropriate n and choose p, q, $p < q$ corresponding to n, then the sets $\mathscr{M}_{p-1}(\{np\})$ and $\mathscr{M}_{q-1}(\{nq\})$ are disjoint. But this is clear since p divides all members of the first set but no members of the second set.

Now we let

$$L_n = \bigcup_{p\in(y,h(n)/(2-h(n))]} \mathscr{M}_{p-1}(\{np\})$$

and show that for n, n', $n \neq n'$, the sets L_n and $L_{n'}$ are disjoint. This is also easy, since the members of the first set have y-smooth part n, while the members of the second set have y-smooth part n'.

It remains to show that each L_n is disjoint from $T_0(z, y)$. We again compare y-smooth parts of the members of each set, which are distinguishable since those of L_n are all deficient, while those of $T_0(z, y)$ are all nondeficient. □

By this lemma, we see that we may add the densities of the sets $\mathscr{M}_{p-1}(\{np\})$. These sets are periodic so we immediately have that

$$\mathbf{d}\,\mathscr{M}_{p-1}(np\mathbb{N}) = \frac{F(p-1)}{np}.$$

We also define their union as

$$\mathscr{A}_{P1} := \bigcup_{\substack{n\le z\\ P(n)\le y\\ 2\left(1-\frac{1}{y+1}\right)\le h(n)<2}} \bigcup_{y<p\le\frac{h(n)}{2-h(n)}} \mathscr{M}_{p-1}(\{np\}),$$

so that we have a new density expression

$$\mathbf{d}\,\mathscr{A}_{P1} = \sum_{\substack{n\le z\\ P(n)\le y\\ 2\left(1-\frac{1}{y+1}\right)\le h(n)<2}} \frac{1}{n} \sum_{y<p\le\frac{h(n)}{2-h(n)}} \frac{F(p-1)}{p} \tag{4.1}$$

which may be added to the medium primes lower bound.

To simplify the calculation of the inner sum we use an observation found in de Bruijn [6].

Lemma 4.3. *For* $0 < y_1 < y_2$,

$$\sum_{p\in(y_1,y_2]} \frac{F(p-1)}{pF(y_1)} = 1 - \frac{F(y_2)}{F(y_1)}. \tag{4.2}$$

Proof. We proceed by induction on the number of primes in $(y_1, y_2]$. For the case where there are no primes in $(y_1, y_2]$, the left-hand sum is empty so is 0, while on the right side of (4.2) we have $F(y_2) = F(y_1)$ so the right side is also 0 and we have equality. Now suppose the equation holds when there are a certain number of primes in $(y_1, y_2]$, and let $(y_2, y_3]$ be an interval containing a single prime p'. Then

$$\begin{aligned}\sum_{p\in(y_1,y_3]} \frac{F(p-1)}{pF(y_1)} &= \sum_{p\in(y_1,y_2]} \frac{F(p-1)}{pF(y_1)} + \frac{F(y_2)}{p'F(y_1)}\\ &= 1 - \frac{F(y_2)}{F(y_1)} + \frac{F(y_2)}{p'F(y_1)}\\ &= 1 - \frac{F(y_2)}{F(y_1)}\left(1-\frac{1}{p'}\right)\\ &= 1 - \frac{F(y_3)}{F(y_1)},\end{aligned}$$

which proves the claim. □

This lemma allows us to simplify the inner sum in (4.1) to

$$\sum_{y<p\le\frac{h(n)}{2-h(n)}} \frac{F(p-1)}{p} = F(y) - F\left(\frac{h(n)}{2-h(n)}\right).$$

Thus, from (4.1),

$$\mathbf{d}\,\mathscr{A}_{P1} = \sum_{\substack{n\le z\\ P(n)\le y\\ 2\left(1-\frac{1}{y+1}\right)\le h(n)<2}} \frac{1}{n}\left(F(y) - F\left(\frac{h(n)}{2-h(n)}\right)\right). \tag{4.3}$$

We will call this the *single large primes lower bound method.*

To implement this sum we must calculate values of $F(y)$ for many values of y. Since $F(y)$ only changes at prime values of y, we include in the initialization of the program the calculation of an array of values $F(p)$ for primes p up to some bound $p_{\max}$. For $y > p_{\max}$, we use Dusart's lower bound for $F(y)$ in [10]. For future reference, we will reproduce both the upper and lower bounds for $F(y)$ here.

$$\frac{e^{-\gamma}}{\log y}\left(1-\frac{0.2}{\log^2 y}\right) \underset{y\geq 2973}{<} F(y) \underset{y>1}{<} \frac{e^{-\gamma}}{\log y}\left(1+\frac{0.2}{\log^2 y}\right). \tag{4.4}$$

Then for each deficient y-smooth n not greater than z, if $h(n) > 2(1-1/(y+1))$, we compute the appropriate term in the sum $\mathbf{d}\,\mathscr{A}_{P1}$.

Using $y = 500$, $z = 10^{14}$, and $p_{\max} = 5 \times 10^7$, we find that

$$\mathbf{d}\,\mathscr{A}_D(z,y) + \mathbf{d}\,\mathscr{A}_{P1} \geq 0.247574757,$$

which is an improvement of about 1.23×10^{-4} over the Deléglise lower bound of 0.247451383. We also have

$$\mathbf{d}\,T_0(z,y) + \mathbf{d}\,\mathscr{A}_{P1} \geq 0.247599114,$$

which is a gain of about 2.43×10^{-5} over the single large primes method alone, and about 1.47×10^{-4} over the Deléglise lower bound.

Remark 4.4. The idea for the medium primes method of Subsection 4.1.2 can be extended using the idea of the single large primes method to give an additional improvement in the density lower bound estimate. We simply consider when a y-smooth n, $z/y < n \leq z$ that is deficient can be augmented by any prime p in the interval $(y_n, y]$ that makes np nondeficient. Then the multiples mnp such that $p(m) \geq p$ have not yet been considered in the single large primes method, since the y-smooth parts of these new numbers are greater than z, while the y-smooth parts of the numbers considered so far in the single large primes method are not greater than z. With this additional contribution to the density, we gain a contribution of about 1.05×10^{-7}. However, as this contribution is marginal, we will not pursue the matter any further.

The double large primes method. Further modifications can be made in the same vein and in addition to the above calculation. The single prime method will not apply for a deficient y-smooth number $n \leq z$ and a prime $p > y$ if $h(np)$ is deficient. Solving the inequality $h(np) < 2$ for each of $h(n)$ and p, we see that this is the case when

$$h(n) < 2 - \frac{2}{p+1} \qquad \text{and, equivalently,} \qquad p > \frac{h(n)}{2 - h(n)}.$$

Since $p > y$, we see that the first inequality always holds when $h(n) \leq 2 - 2/(y+1)$. We will call this case I. On the other hand, even if $2 - 2/(y+1) < h(n) < 2$, it may still be true that $p > h(n)/(2 - h(n))$. This will be case II.

In either case I or case II, there may be a prime $q > p$ such that npq is abundant. Then analogously as before we have for a deficient number np that the sum of these densities can be simplified:

$$\frac{1}{np} \sum_{q \in (p, h(np)/(2-h(np))]} \frac{F(q-1)}{q} = \frac{1}{np} \left(F(p) - F\left(\frac{h(np)}{2 - h(np)} \right) \right). \tag{4.5}$$

This density represents the multiples $mnpq$ where $p(m) \geq q$.

We now note that, just as in the case of the small primes method of Subsection 4.1.1, the idea can be repeated for higher powers of p. For instance, if the number npq defined as above is abundant, then np^2q is also abundant, so repeating the previous argument we have the density sum

$$\frac{1}{np^2} \sum_{q \in (p, h(np)/(2-h(np))]} \frac{F(q-1)}{q}.$$

These sets are disjoint from the sets involving npq since the numbers differ in their powers of p. Moreover, we can continue to higher powers of p, giving the density sums

$$\frac{1}{np^i} \sum_{q \in (p, h(np)/(2-h(np))]} \frac{F(q-1)}{q}$$

for all $i \geq 1$. The sum of these densities over $i \geq 1$ is a geometric series, so we simply multiply (4.5) by a factor of $p/(p-1)$. The sum thus becomes

$$\frac{1}{n(p-1)} \left(F(p) - F\left(\frac{h(np)}{2-h(np)} \right) \right). \tag{4.6}$$

It can be verified that this equals the density of the union of the sets involved by noting that the union of the tail is bounded by a geometric series and thus has limit zero.

We now evaluate and sum over this expression for each prime $p > y$ which is accompanied by a second prime $q > p$ in the interval

$$a < p < q \leq \frac{h(np)}{2-h(np)},$$

where $a = y$ for case I, and $a = h(n)/(2-h(n))$ for case II.

Noting that if we allow p to be the largest prime satisfying $p \leq h(np)/(2-h(np))$, the density expression (4.6) is 0, so we may simply sum over primes

$$a < p \leq \frac{h(np)}{2 - h(np)}.$$

By solving the second inequality above for p, we find an upper bound for p independent of p, namely

$$\begin{aligned} p &\leq \frac{h(n)}{2-h(n)} + \sqrt{\left(\frac{h(n)}{2-h(n)}\right)^2 + \frac{h(n)}{2-h(n)}} \\ &= \frac{h(n)}{2-h(n)}\left(1 + \sqrt{\frac{2}{h(n)}}\right) \\ &= \frac{\sqrt{h(n)}}{\sqrt{2} - \sqrt{h(n)}}. \end{aligned}$$

Using this bound on p, we may now determine a lower bound for $h(n)$ in case I. We find that

$$h(n) \geq 2\left(1 - \frac{1}{p+1}\right)^2,$$

and since $p > y$, this gives us

$$h(n) \geq 2\left(1 - \frac{1}{y+1}\right)^2 = 2 - \frac{4}{y+1} + \frac{2}{(y+1)^2}.$$

We will denote by $\mathscr{A}_{P2I}$ and $\mathscr{A}_{P2II}$ the set of nondeficient numbers belonging to case I and II, respectively, so that

$$\mathscr{A}_{P2I} := \bigcup_{\substack{n \leq z \\ P(n) \leq y}} \bigcup_{\substack{h(n) \leq 2 - \frac{2}{y+1} \\ y < p < q \leq \frac{h(np)}{2-h(np)}}} \bigcup_{i=1}^{\infty} \mathscr{M}_{q-1}(\{np^i q\})$$

and

$$\mathscr{A}_{P2II} := \bigcup_{\substack{n\le z\\ P(n)\le y}} \bigcup_{\substack{2-\frac{2}{y+1}<h(n)<2\\ \frac{h(n)}{2-h(n)}<p<q\le\frac{h(np)}{2-h(np)}}} \bigcup_{i=1}^{\infty} \mathscr{M}_{q-1}(\{np^i q\}).$$

We will in addition define $\mathscr{A}_{P2} := \mathscr{A}_{P2I} \cup \mathscr{A}_{P2II}$. Then $\mathbf{d}\,\mathscr{A}_{P2}$ is the sum of terms (4.6) over n and p satisfying the conditions of either case I or case II, so that

$$\mathbf{d}\,\mathscr{A}_{P2} = \sum_{\substack{n\le z\\ P(n)\le y\\ b_2<h(n)\le b_1}} \frac{1}{n} \sum_{y<p\le a_2} \frac{1}{p-1}\left(F(p) - F\left(\frac{h(np)}{2-h(np)}\right)\right) + \sum_{\substack{n\le z\\ P(n)\le y\\ b_1\le h(n)<2}} \frac{1}{n} \sum_{a_1<p\le a_2} \frac{1}{p-1}\left(F(p) - F\left(\frac{h(np)}{2-h(np)}\right)\right), \quad (4.7)$$

where $a_i = h(n)^{1/i}/(2^{1/i} - h(n)^{1/i})$ and $b_i = 2(1-1/(y+1))^i$.

At this point another simplification may be made. Each of the inner sums in (4.7) can be split into two sums,

$$\sum_{a<p\le a_2} \frac{F(p)}{p-1} \qquad \text{and} \qquad -\sum_{a<p\le a_2} \frac{1}{p-1} F\left(\frac{h(np)}{2-h(np)}\right).$$

We now observe that by (4.2), the first sum can be written as

$$\sum_{a<p\le a_2} \frac{F(p)}{p-1} = \sum_{a<p\le a_2} \frac{F(p-1)}{p-1}\cdot\frac{p-1}{p} = F(a) - F(a_2).$$

Then we have a computationally simpler expression for $\mathbf{d}\,\mathscr{A}_{P2}$, namely

$$\mathbf{d}\,\mathscr{A}_{P2} = \sum_{\substack{n\le z\\ P(n)\le y\\ b_2<h(n)\le b_1}} \frac{1}{n}\left(F(y)-F(a_2)-\sum_{y<p\le a_2}\frac{1}{p-1}F\left(\frac{h(np)}{2-h(np)}\right)\right)+$$

$$\sum_{\substack{n\le z\\ P(n)\le y\\ b_1\le h(n)<2}} \frac{1}{n}\left(F(a_1)-F(a_2)-\sum_{a_1<p\le a_2}\frac{1}{p-1}F\left(\frac{h(np)}{2-h(np)}\right)\right). \quad (4.8)$$

For computational purposes we may either limit the largest possible prime p to be bounded by some $p_{\max}$, and compute terms only when $a_2 \le p_{\max}$ so that $F(a_2)$ may be computed, or when $a_2 > p_{\max}$ we may use explicit upper and lower bounds on $F(x)$ which may be found, for instance, in [10].

By using the latter approach, we can use a combined expression for $\mathbf{d}\,\mathscr{A}_{P1}+\mathbf{d}\,\mathscr{A}_{P2}$ by adding Equations (4.3) and (4.8),

$$\mathbf{d}\,\mathscr{A}_{P1}+\mathbf{d}\,\mathscr{A}_{P2} = \sum_{\substack{n\le z\\ P(n)\le y\\ b_2<h(n)<2}} \frac{1}{n}\left(F(y)-F(a_2)-\sum_{\max\{y,a_1\}<p\le a_2}\frac{1}{p-1}F\left(\frac{h(np)}{2-h(np)}\right)\right)$$

$$-\sum_{\substack{n\le z\\ P(n)\le y\\ b_2<h(n)\le b_1}} \frac{1}{n}\sum_{y<p\le a_1}\frac{1}{p-1}F\left(\frac{h(np)}{2-h(np)}\right). \quad (4.9)$$

Finally, we note that

$$y < a_1 \qquad \Longleftrightarrow \qquad h(n) > b_1,$$

so that the second outer sum in (4.9) is in fact empty. Thus we have

$$\mathbf{d}\,\mathscr{A}_{P1} + \mathbf{d}\,\mathscr{A}_{P2} = \sum_{\substack{n\le z \\ P(n)\le y \\ b_2<h(n)<2}} \frac{1}{n}\left(F(y) - F(a_2) - \sum_{\max\{y,a_1\}<p\le a_2} \frac{1}{p-1} F\left(\frac{h(np)}{2-h(np)}\right)\right). \tag{4.10}$$

In practice, we must control the inner sum in (4.10) when the primes p exceed $p_{\max}$, since a_2 can be much larger than $p_{\max}$. This can be seen since $a_2 > 2a_1$, and a_1 can become about as large as $2z$ since there can be n such that $2 - h(n) \approx 1/z$. This happens, for instance, when $n = 2^k$, where k is such that $2^k \le z < 2^{k+1}$. For this choice of n, $a_1 = 2^{k+1} - 1 > z - 1$ so $a_2 > 2z - 2$. Thus when z is large, say $z = 10^{14}$, it is impractical to sum over primes in the interval $(a_1, a_2]$. Moreover, when p is as large as z, the contribution of the term corresponding to p is smaller than $1/z$, so can be ignored with miniscule cost.

To handle this issue, if $p > p_{\max}$ we bound the terms of the inner sum in (4.10) by first observing that

$$\frac{h(np)}{2-h(np)} \ge a_2 \qquad \Longleftrightarrow \qquad p \le a_2.$$

Thus

$$F\left(\frac{h(np)}{2-h(np)}\right) \le F(a_2).$$

We also use the bound

$$\begin{aligned} \sum_{p\in(a,b]} \frac{1}{p-1} &= \sum_{p\in(a,b]} \left(\frac{1}{p} + \frac{1}{p(p-1)}\right) \\ &\le \sum_{p\in(a,b]} \frac{1}{p} + \sum_{n\in(a,b]} \frac{1}{n(n-1)} \end{aligned}$$

$$= \sum_{p\in(a,b]} \frac{1}{p} + \sum_{n\in(a,b]} \left(\frac{1}{n-1} - \frac{1}{n}\right)$$
$$= \sum_{p\in(a,b]} \frac{1}{p} + \frac{1}{a} - \frac{1}{b}.$$

Then we have the bound

$$\sum_{\max\{a_1,p_{\max}\}<p\leq a_2} \frac{1}{p-1} F\left(\frac{h(np)}{2-h(np)}\right)$$
$$\leq F(a_2)\left(\sum_{\max\{a_1,p_{\max}\}<p\leq a_2} \frac{1}{p} + \frac{1}{\max\{a_1,p_{\max}\}} - \frac{1}{a_2}\right).$$

We can then use Dusart's upper bound for the sum over reciprocal primes from [10] to bound

$$\sum_{\max\{a_1,p_{\max}\}<p\leq a_2} \frac{1}{p}.$$

For future reference, we will record Dusart's upper and lower bounds for the sum of reciprocal primes here:

$$-\left(\frac{1}{10\log^2 x} + \frac{4}{15\log^3 x}\right) \underset{x>1}{\leq} \sum_{p\leq x} \frac{1}{p} - \log\log x - B \underset{x\geq 10372}{\leq} \frac{1}{10\log^2 x} + \frac{4}{15\log^3 x}, \quad (4.11)$$

where B is defined by the sum over primes p,

$$B = \gamma + \sum_p \left(\log\left(1-\frac{1}{p}\right) + \frac{1}{p}\right),$$

and γ is Euler's constant.

We will call this method of using primes $p > y$ to augment the Deléglise lower bound calculation for the density of abundant numbers the *large primes method.*

Recall that the single prime version of the large primes method with $y = 500$, $z = 10^{14}$, and $p_{\max} = 5 \times 10^7$ gave us the bound

$$\mathbf{d}\,\mathscr{A} \geq 0.247574758,$$

which is an improvement of about 1.23×10^{-4} over the Deléglise lower bound of 0.247451383. If we also use the double prime version of the large primes method, we improve this to

$$\mathbf{d}\,\mathscr{A} \geq 0.247592145,$$

which improves on the single prime result by about 1.73×10^{-5}.

By combining the medium primes method together with the large primes method, we can further improve the lower bound given by Deléglise. In particular, we have the following theorem.

Theorem 4.5. *The density of abundant numbers can be bounded below by*

$$\mathbf{d}\,\mathscr{A} \geq \mathbf{d}\,T_0(z, y) + \mathbf{d}\,\mathscr{A}_{P1} + \mathbf{d}\,\mathscr{A}_{P2}.$$

In particular,

$$\mathbf{d}\,\mathscr{A} \geq 0.247616464,$$

which is an improvement of approximately 1.65×10^{-4} *over the value of the lower bound found by Deléglise.*

Proof. As noted earlier, the subsets of abundant numbers considered in each of these new methods are disjoint from $\mathscr{A}_D(z, y)$. In fact these new sets are also disjoint to each other, as can be seen by comparing the y-smooth parts of the members. In addition, note that for the medium primes method we consider numbers having y-smooth part abundant, while the large primes method considers numbers having y-smooth part deficient. This proves the lower bound expression.

Using the various methods with parameters $y = 500$, $z = 10^{14}$, and $p_{\max} = 5 \times 10^7$, we calculate the stated value for the lower bound expression. □

4.2 Strategies for upper bound improvements

Recall the Behrend-Deléglise upper bound method of Section 3.1. We reproduce the upper bound expression (3.5) here:

$$\mathbf{d}\,\mathscr{A}_\alpha \le \sum_{\substack{n \le z \\ P(n) \le y}} \frac{\tilde{A}_{y,\alpha/h(n)}}{n} + 1 - F(y) \sum_{\substack{n \le z \\ P(n) \le y}} \frac{1}{n}.$$

The first sum involves the expression $\tilde{A}_{y,\alpha}$, which is an upper bound for the density $\mathbf{d}\,\mathscr{A}_{y,\alpha}$. Thus one strategy for improving the Behrend-Deléglise upper bound method is to improve the upper bound on $\mathbf{d}\,\mathscr{A}_{y,\alpha}$.

We will also investigate the upper bound analogues of the large primes method of Subsection 4.1.3. As these methods also rely on $\mathbf{d}\,\mathscr{A}_{y,\alpha}$, the improvements in bounding $\mathbf{d}\,\mathscr{A}_{y,\alpha}$ will carry over to improvements in the upper bound version of the large primes method as well.

4.2.1 Improving bounds on $\mathbf{d}\,\mathscr{A}_{y,\alpha}$

The Behrend moment method of Proposition 2.16 bounds $\mathbf{d}\,\mathscr{A}_{y,\alpha}$ by using moments $M_r(h_y)$ of h_y, as defined in Section 2.3. If we instead make explicit an asymptotic estimate for $\mathbf{d}\,\mathscr{A}_{y,\alpha}$ as $\alpha \to 1^+$ we can directly find both upper and lower bounds for the density. We will also modify the original moment method to find an improved upper bound for $\mathbf{d}\,\mathscr{A}_{y,\alpha}$. This will require first calculating to many decimal digits the moments of h_y.

Using asymptotic estimates for $\mathbf{d}\,\mathscr{A}_{y,\alpha}$

Recall the definition of $\mathbf{d}\,\mathscr{A}_{y,\alpha}$ from Section 2.3. The Behrend moment method does not do well when α is near 1. In fact, for α sufficiently close to 1, the trivial bound $F(y)$ is used. We can see this by comparing the two bounds:

$$F(y) \leq F(y)\frac{M_r(h_y)-1}{\alpha^r-1} \qquad \Longleftrightarrow \qquad \alpha \leq M_r(h_y).$$

Since $M_r(h_y)$ increases with r, the trivial bound is better than the moment method when $1 < \alpha < M_1(h_y)$.

However, by studying the asymptotic behavior of $\mathbf{d}\,\mathscr{A}_{y,\alpha}$ as $\alpha \to 1^+$, we know that the trivial bound $F(y)$ is far from the actual value of $\mathbf{d}\,\mathscr{A}_{y,\alpha}$ in the interval $(1, M_1(h_y))$. By making the asymptotic estimate explicit, we are able to find a non-trivial upper bound for values of α in this region.

The asymptotic behavior of the function $\mathbf{d}\,\mathscr{A}_{\alpha}$ has been studied for the case $\alpha \to$

1^+. In particular, Erdős found in [13] that as $\epsilon \to 0^+$,

$$1 - \mathbf{d}\,\mathscr{A}_{1+\epsilon} = (1+o(1))\frac{e^{-\gamma}}{\log \epsilon^{-1}}.$$

By using known explicit bounds on the distribution of primes, and applying these to the proof of the asymptotic result, we can find an upper bound for $\mathbf{d}\,\mathscr{A}_\alpha$. An analogous argument applies for $\mathbf{d}\,\mathscr{A}_{y,\alpha}$.

Explicit Erdős bounds

We will determine explicit upper and lower bounds on $\mathbf{d}\,\mathscr{A}_{y,1+\epsilon}$ for $y \le \epsilon^{-1}$. Let n be an integer with $h(n) < 1+\epsilon$. Note that n is not divisible by any prime $q \le \epsilon^{-1}$, since if it were, then

$$h(n) \ge h(q) = 1 + \frac{1}{q} \ge 1 + \epsilon,$$

a contradiction. Thus for any prime $p \le \epsilon^{-1}$ we have

$$\{n : h(n) < 1+\epsilon\} \subseteq \{n : (n, \Pi(\epsilon^{-1})) = 1\}.$$

This gives us the inequality of their densities,

$$1 - \mathbf{d}\,\mathscr{A}_{1+\epsilon} \le F(\epsilon^{-1}).$$

Since

$$\{n : (n, \Pi(y)) = 1, h(n) < 1+\epsilon\} \subseteq \{n : h(n) < 1+\epsilon\},$$

we also get the density relation

$$F(y) - \mathbf{d}\,\mathscr{A}_{y,1+\epsilon} \le F(\epsilon^{-1}). \tag{4.12}$$

Note that since $y \le \epsilon^{-1}$, $F(y) \ge F(\epsilon^{-1})$, and we have found a nontrivial inequality. Solving the inequality (4.12) for $\mathbf{d}\,\mathscr{A}_{y,1+\epsilon}$ gives us the lower bound for $y \le \epsilon^{-1}$ of

$$F(y) - F(\epsilon^{-1}) \le \mathbf{d}\,\mathscr{A}_{y,1+\epsilon}. \tag{4.13}$$

We next work on determining an upper bound for $\mathbf{d}\,\mathscr{A}_{y,1+\epsilon}$ when $2 \le y \le \epsilon^{-1}$. First we determine a property of numbers n not divisible by any primes $p \le \epsilon^{-1}$ such that $h(n) \ge 1+\epsilon$. In particular we will show that such numbers n must have for some positive integer t at least t prime factors in the interval $J_t = (4^{t-1}\epsilon^{-1}, 4^t\epsilon^{-1}]$. We will call this property on n property A. We will say that a number n has property $\mathrm{A_t}$ if it has at least t prime factors in J_t for a particluar t.

Suppose n does not have property A. Then

$$\begin{aligned}
\frac{\sigma(n)}{n} < \frac{n}{\phi(n)} = \prod_{q|n} \frac{q}{q-1} &< \prod_{t=1}^{\infty} \prod_{\substack{q \in J_t \\ q|n}} \frac{q}{q-1} \\
&< \exp \sum_{t=1}^{\infty} \sum_{\substack{q \in J_t \\ q|n}} \log\left(1 + \frac{1}{q-1}\right) \\
&< \exp \sum_{t=1}^{\infty} \frac{t-1}{4^{t-1}\epsilon^{-1} - 1}.
\end{aligned}$$

Next we use the bound $2 \leq \epsilon^{-1}$ so that

$$\begin{aligned}
\exp \sum_{t=1}^{\infty} \frac{t-1}{4^{t-1}\epsilon^{-1} - 1} &< \exp \epsilon \sum_{t=1}^{\infty} \frac{t-1}{4^{t-1}} \left(1 + \frac{2\epsilon}{4^{t-1}}\right) \\
&\leq \exp \epsilon \sum_{t=1}^{\infty} \frac{t-1}{4^{t-1}} \left(1 + \frac{1}{4^{t-1}}\right) \\
&= \exp \frac{116}{225}\epsilon,
\end{aligned}$$

using

$$\sum_{t \geq 1} (t-1)\alpha^{t-1} = \frac{\alpha}{(1-\alpha)^2}$$

for $|\alpha| < 1$.

We now use the estimate

$$\exp x \leq 1 + 2(e^{1/2} - 1)x < 1 + 1.3x$$

for $0 \leq x \leq 1/2$ to get

$$h(n) < \exp \frac{116}{225}\epsilon < 1 + 0.7\epsilon.$$

This contradicts our hypothesis, establishing that property A holds for n.

Next we estimate the density of the numbers satisfying property $\mathrm{A_t}$. Let the integers $a_1, a_2, \ldots, a_l$ denote the numbers with exactly t prime factors from J_t. Then the density of multiples $a_i m$ with $(m, \Pi(\epsilon^{-1})) = 1$ is

$$\frac{F(\epsilon^{-1})}{a_i}.$$

Thus, the density of integers containing at least t prime factors from J_t, namely those having property $\mathrm{A_t}$, is bounded by

$$F(\epsilon^{-1})\sum_{i=1}^{l}\frac{1}{a_i}.$$

By the multinomial theorem, we have

$$\sum_{i=1}^{l}\frac{1}{a_i}\leq\frac{\left(\sum_{p\in J_t}\frac{1}{p}\right)^t}{t!}.$$

Calling $\mathscr{S}_t$ the set of numbers having property $\mathrm{A_t}$, we thus have

$$\mathbf{d}\,\mathscr{S}_t\leq F(\epsilon^{-1})\frac{\left(\sum_{p\in J_t}\frac{1}{p}\right)^t}{t!}.$$

Summing this inequality over all t, we assert that

$$\mathbf{d}\bigcup_{t=1}^{\infty}\mathscr{S}_t\leq F(\epsilon^{-1})\sum_{t=1}^{\infty}\frac{\left(\sum_{p\in J_t}\frac{1}{p}\right)^t}{t!}. \tag{4.14}$$

To prove this, we must show that

$$\lim_{t_0\to\infty}\mathbf{d}\bigcup_{t=t_0}^{\infty}\mathscr{S}_t=0.$$

In fact, since

$$\sum_{p\leq x}\frac{1}{p}=\log\log x+O(1),$$

we have

$$\sum_{p\in J_t}\frac{1}{p} = \log\log(4^t\epsilon^{-1}) - \log\log(4^{t-1}\epsilon^{-1}) + O(1)$$
$$= \log\left(1-\frac{\log 4}{\log 4+\log(4^{t-1}\epsilon^{-1})}\right) + O(1)$$
$$= O(1)$$

for $t \geq 1$. Thus the sequence

$$\sum_{p\in J_t}\frac{1}{p} \tag{4.15}$$

in t is bounded, say by some bound C, so

$$\mathbf{d}\bigcup_{t=t_0}^{\infty}\mathscr{S}_t \leq F(\epsilon^{-1})\sum_{t=t_0}^{\infty}\frac{C^t}{t!}.$$

Since the sum on the right side is the tail of the Maclaurin series for e^C, which converges, we have shown that the bound (4.14) holds.

Since the set of numbers satisfying property $\mathrm{A_t}$ for each t contains the numbers n such that $h(n) \geq 1+\epsilon$ and $(n, \Pi(\epsilon^{-1})) = 1$, we have

$$\mathbf{d}\,\mathscr{A}_{\epsilon^{-1},1+\epsilon} \leq \mathbf{d}\bigcup_{t=t_0}^{\infty}\mathscr{S}_t.$$

In fact, since $2 \leq y \leq \epsilon^{-1}$, we have by (4.14) the upper bound

$$\mathbf{d}\,\mathscr{A}_{y,1+\epsilon} \leq F(\epsilon^{-1})\sum_{t=1}^{\infty}\frac{\left(\sum_{p\in J_t}\frac{1}{p}\right)^t}{t!}. \tag{4.16}$$

In practice, we set up an array for the sum of reciprocal primes for primes up to

$p_{\max}$ so that we may calculate (4.15) for t satisfying $4^t\epsilon^{-1} < p_{\max}$. For larger t, we may use Dusart's upper bound (4.11) for the sum of reciprocal primes.

This gives us a bound for (4.15) for t when $4^t\epsilon^{-1} \geq p_{\max}$, provided $10372 \leq p_{\max}$. The bound is

$$\begin{aligned}
\sum_{p\in J_t} \frac{1}{p} &< \log\log 4^t\epsilon^{-1} + \frac{1}{10\log^2 4^t\epsilon^{-1}} + \frac{4}{15\log^3 4^t\epsilon^{-1}} \\
&\quad - \log\log 4^{t-1}\epsilon^{-1} + \frac{1}{10\log^2 4^{t-1}\epsilon^{-1}} + \frac{4}{15\log^3 4^{t-1}\epsilon^{-1}} \\
&= \log\left(\frac{\log 4^t\epsilon^{-1}}{\log 4^{t-1}\epsilon^{-1}}\right) + \frac{1}{10\log^2 4^t\epsilon^{-1}} + \frac{4}{15\log^3 4^t\epsilon^{-1}} \\
&\quad + \frac{1}{10\log^2 4^{t-1}\epsilon^{-1}} + \frac{4}{15\log^3 4^{t-1}\epsilon^{-1}} \\
&= \log\left(1 + \frac{\log 4}{\log 4^{t-1}\epsilon^{-1}}\right) + \frac{1}{10\log^2 4^t\epsilon^{-1}} + \frac{4}{15\log^3 4^t\epsilon^{-1}} \\
&\quad + \frac{1}{10\log^2 4^{t-1}\epsilon^{-1}} + \frac{4}{15\log^3 4^{t-1}\epsilon^{-1}} =: U_t.
\end{aligned}$$

From the final expression it is clear that as t increases, U_t decreases, so we can calculate the initial terms of the sum for t up to some bound T, and then estimate the tail of the sum as an exponential, as follows:

$$\sum_{t=1}^{\infty} \frac{\left(\sum_{q\in J_t}\frac{1}{q}\right)^t}{t!} < \sum_{t=1}^{T} \frac{\left(\sum_{q\in J_t}\frac{1}{q}\right)^t}{t!} + \exp(U_{T+1}) - \sum_{t=0}^{T}\frac{U_{T+1}^t}{t!}.$$

Thus the density $\mathbf{d}\,\mathscr{A}_{\epsilon^{-1},1+\epsilon}$ has upper bound

$$\mathbf{d}\,\mathscr{A}_{\epsilon^{-1},1+\epsilon} \leq F(\epsilon^{-1})\left(\sum_{t=1}^{T} \frac{\left(\sum_{q\in J_t}\frac{1}{q}\right)^t}{t!} + \exp(U_{T+1}) - \sum_{t=0}^{T}\frac{U_{T+1}^t}{t!}\right) =: \overline{A}_{\epsilon^{-1},1+\epsilon}. \tag{4.17}$$

We now use that for $p \le \epsilon^{-1}$,

$$F(p) - \mathbf{d}\,\mathscr{A}_{p,1+\epsilon} = F(\epsilon^{-1}) - \mathbf{d}\,\mathscr{A}_{\epsilon^{-1},1+\epsilon}. \tag{4.18}$$

This comes from the observation that the sets which the densities represent are equal:

$$\{n : (n, \Pi(p)) = 1, h(n) < 1+\epsilon\} = \{n : (n, \Pi(\epsilon^{-1})) = 1, h(n) < 1+\epsilon\},$$

which can be seen by the following observation. If n is in the left hand set, $h(n) < 1+\epsilon$, and no prime $q \le p \le \epsilon^{-1}$ can divide n since otherwise $h(n) \ge h(q) \ge 1+\epsilon$, a contradiction. Thus n is in the right hand set. Likewise if n is in the right hand set, $h(n) < 1+\epsilon$, and since $p \le \epsilon^{-1}$, it cannot divide n. Thus the two sets in question are equal.

Using equation (4.18) with inequality (4.17), we arrive at the bound

$$\begin{aligned} \mathbf{d}\,\mathscr{A}_{p,1+\epsilon} &= F(p) - F(\epsilon^{-1}) + \mathbf{d}\,\mathscr{A}_{\epsilon^{-1},1+\epsilon} \\ &\le F(p) - F(\epsilon^{-1}) + \overline{A}_{\epsilon^{-1},1+\epsilon}. \end{aligned}$$

Thus, together with (4.13) and (4.17), we have the following theorem.

Theorem 4.6. *For prime $p \le \epsilon^{-1}$,*

$$F(p) - F(\epsilon^{-1}) \le \mathbf{d}\,\mathscr{A}_{p,1+\epsilon} \le F(p) - F(\epsilon^{-1}) + \overline{A}_{\epsilon^{-1},1+\epsilon},$$

where

$$\overline{A}_{\epsilon^{-1},1+\epsilon} := F(\epsilon^{-1}) \left(\sum_{t=1}^{T} \frac{\left(\sum_{q \in J_t} \frac{1}{q}\right)^t}{t!} + \exp(U_{T+1}) - \sum_{t=0}^{T} \frac{U_{T+1}^t}{t!} \right).$$

The base 4 in property A, which was taken directly from Erdős' paper [13], is a convenient integral value to use. However, we can also replace this base in the interval J_t by a more optimal base. Suppose we call this base c, so that we are considering the interval $J_t' = [c^{t-1}\epsilon^{-1}, c^t\epsilon^{-1})$. The property corresponding to property A, which we call property B, only holds for certain values of c. Tracing through our argument, we find that c must satisfy the condition

$$2(\sqrt{e}-1)\frac{c(c^2+3c+1)}{(c^2-1)^2} < 1.$$

The smallest we can take c is around 3.222.

In practice, the value T in Theorem 4.18 is chosen to be the largest value that can be used given our posession of primes up to $p_{\max}$. Namely, we let T be the largest t such that $c^t\epsilon^{-1} \le p_{\max}$.

We will illustrate the effectiveness of this method by comparing it with the original method at a certain value of ϵ. Neither the Deléglise upper bound method nor the reduced moment upper bound method perform well when $1+\epsilon$ is near $M_1(h_y)$. For instance, each of these first moment methods gives the trivial upper bound $F(y)$ at $1+\epsilon = M_1(h_y)$. In contrast, we see a nontrivial improvement when we use the asymptotic method of this section at this value. We find with $c = 4$ that

$$\mathbf{d}\,\mathscr{A}_{500,M_1(h_{500})} \le 0.0330949555,$$

as opposed to the trivial bound

$$\mathbf{d}\,\mathscr{A}_{500,M_1(h_{500})} \leq F(500) = 0.0896097368\ldots.$$

Adjusting the value of c to 3.222, we see an improvement to

$$\mathbf{d}\,\mathscr{A}_{500,M_1(h_{500})} \leq 0.0306312737.$$

We may compare this value to the corresponding lower bound,

$$\mathbf{d}\,\mathscr{A}_{500,M_1(h_{500})} \geq F(500) - F(\epsilon^{-1}) = 0.0213253662.$$

The ζ-factor method

For Deléglise's program, upper bounds are calculated for the moments $M_r(h_y)$ of h_y in the following manner. First the moments are expressed as Euler products, and then upper bounds are determined for each factor, treating the case of large and small primes separately. In general this method does not determine $M_r(h_y)$ to high precision. By using a different approach, we will find estimates of $M_r(h_y)$ for the first few values of r that are correct to many decimal places.

This approach, which we will call the *ζ-factor method*, begins with the Euler product of $M_r(h_y)$ and accelerates the convergence of this product by expressing this Euler product in terms of products of $\zeta(n)$ and a remainder factor which converges quickly. The factor involving $\zeta(n)$ can be quickly computed, using, for instance, the computer program PARI. This program calculates $\zeta(n)$ for even n using Bernoulli numbers, while for odd n modular forms are used.

Recall that the Euler product representation of $M_r(h_y)$ is given by

$$M_r(h_y) = \prod_{p>y}\left(1 + \frac{\rho(p)}{p} + \frac{\rho(p^2)}{p^2} + \cdots\right) = \prod_{p>y}(1 + s(p,r))$$

where $s(p,r) = \frac{\rho(p)}{p} + \frac{\rho(p^2)}{p^2} + \cdots$ and

$$\rho(p^\alpha) = \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^\alpha}\right)^r - \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{\alpha-1}}\right)^r.$$

As it stands, $s(p,r)$ is an infinite series. It would be convenient to find a closed form expression for calculations. One such closed form can be found as follows.

$$\begin{aligned}
s(p,r) &= \frac{\left(1+\frac{1}{p}\right)^r - 1^r}{p} + \frac{\left(1+\frac{1}{p}+\frac{1}{p^2}\right)^r - \left(1+\frac{1}{p}\right)^r}{p^2} + \cdots \\
&= \frac{1}{p}\sum_{i=1}^{r}\binom{r}{i}\left(\frac{1}{p}\right)^i + \frac{1}{p^2}\sum_{i=1}^{r}\binom{r}{i}\left(1+\frac{1}{p}\right)^{r-i}\left(\frac{1}{p^2}\right)^i + \cdots \\
&= \sum_{n=1}^{\infty}\frac{1}{p^n}\sum_{i=1}^{r}\binom{r}{i}\left(\sum_{j=0}^{n-1}\frac{1}{p^j}\right)^{r-i}\left(\frac{1}{p^n}\right)^i \\
&= \sum_{i=1}^{r}\binom{r}{i}\sum_{n=1}^{\infty}\frac{1}{p^n}\left(\frac{\frac{1}{p^n}-1}{\frac{1}{p}-1}\right)^{r-i}\left(\frac{1}{p^n}\right)^i \\
&= \sum_{i=1}^{r}\binom{r}{i}\left(\frac{p}{p-1}\right)^{r-i}\sum_{n=1}^{\infty}\left(1-\frac{1}{p^n}\right)^{r-i}\left(\frac{1}{p^n}\right)^{i+1} \\
&= \sum_{i=1}^{r}\binom{r}{i}\left(\frac{p}{p-1}\right)^{r-i}\sum_{n=1}^{\infty}\sum_{j=0}^{r-i}\binom{r-i}{j}(-1)^j\left(\frac{1}{p^n}\right)^j\left(\frac{1}{p^n}\right)^{i+1} \\
&= \sum_{i=1}^{r}\binom{r}{i}\left(\frac{p}{p-1}\right)^{r-i}\sum_{j=0}^{r-i}\binom{r-i}{j}(-1)^j\sum_{n=1}^{\infty}\frac{1}{p^{n(i+j+1)}} \\
&= \sum_{i=1}^{r}\binom{r}{i}\left(\frac{p}{p-1}\right)^{r-i}\sum_{j=0}^{r-i}\binom{r-i}{j}(-1)^j\frac{1}{p^{i+j+1}-1}.
\end{aligned}$$

Remark 4.7. Behrend in [4] gives the alternative expression

$$1 + s(p,r) = \frac{p^r}{(p-1)^{r-1}} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{1}{p^{i+1}-1}.$$

We will not be using this expression in what follows.

In particular, we find for $r = 1$,

$$s(p,1) = \frac{1}{p^2-1},$$

for $r = 2$,

$$s(p,2) = \frac{2p}{(p-1)(p^2-1)} - \frac{p+1}{(p-1)(p^3-1)} = \frac{2p^3+p^2-1}{(p-1)^2(p+1)(p^2+p+1)},$$

for $r = 3$,

$$s(p,3) = \frac{3p^6+p^4-p^3-p+1}{(p-1)(p^3-1)(p^4-1)},$$

and for $r = 4$,

$$s(p,4) = \frac{4p^{11}+2p^{10}+2p^9+p^8+3p^7-2p^6+p^5+p^3+p-1}{(p-1)(p^3-1)(p^4-1)(p^5-1)}.$$

We now return our attention to the Euler products. Note that for $r = 1$ and $y = 1$,

$$M_1(h_1) = \prod_p \left(1 + \frac{1}{p^2-1}\right) = \zeta(2).$$

We can find the value of $\zeta(2)$ using the computer program PARI, which, in this case amounts to calculating the value of $\pi^2/6$. Then using this value as a starting point, we may calculate $M_1(h_y)$ for larger values of y by dividing $\zeta(2)$ by

$$\prod_{p \le y} \left(1 + \frac{1}{p^2 - 1}\right),$$

since the latter is a finite calculation.

For the case of $r = 2$, we note that

$$1 + s(p, 2) = 1 + \frac{2p^3 + p^2 - 1}{(p-1)^2(p+1)(p^2+p+1)} = 1 + \frac{2}{p^2}(1 + o(1)).$$

Since

$$\left(1 + \frac{1}{p^2 - 1}\right)^2 = 1 + \frac{2}{p^2}(1 + o(1)),$$

we may factor this term out of $1 + s(p, 2)$, resulting in the factorization

$$1 + \frac{2p^3 + p^2 - 1}{(p-1)^2(p+1)(p^2+p+1)} = \left(1 + \frac{1}{p^2 - 1}\right)^2 \left(1 + \frac{c}{p^3}(1 + o(1))\right).$$

By using a computer algebra system such as Maple, we find that $c = 1$. We thus continue this factorization process by factoring out

$$1 + \frac{1}{p^3 - 1}.$$

At the next step we find that what remains is the factor

$$1 - \frac{1}{p^4},$$

which we recognize as a term of the Euler product for $1/\zeta(4)$. Thus in the case $r = 2$, we have found that

$$\prod_p (1 + s(p,2)) = \frac{\zeta(2)^2\zeta(3)}{\zeta(4)}.$$

We will call the above method of factoring out successive terms of $\zeta(s)$ the ζ-factor method. By applying the ζ-factor method to the case $r = 3$, we find that

$$\prod_p (1 + s(p,3)) = \frac{\zeta(2)^3\zeta(3)^3}{\zeta(4)^2} \prod_p \left(1 - \frac{P_3(p)}{Q_3(p)}\right),$$

where $P_3(x) = 3x^4 + x^3 + 3x^2 + 1$ and $Q_3(x) = x^3(x^2+1)^3$. For $r = 4$, we find

$$\prod_p (1 + s(p,4)) = \frac{\zeta(2)^4\zeta(3)^6}{\zeta(4)^2} \prod_p \left(1 - \frac{P_4(p)}{Q_4(p)}\right),$$

where

$$\begin{aligned} P_4(x) = {} & 11x^{18} + 22x^{17} + 31x^{16} + 16x^{15} - 3x^{14} - 18x^{13} - 19x^{12} - 19x^{11} \\ & - 19x^{10} - 4x^9 + 12x^8 + 27x^7 + 15x^6 + 5x^5 - 7x^4 - 4x^3 - 4x^2 - x - 1 \end{aligned}$$

and

$$Q_4(x) = x^{13}(x^2+1)^3(x^4 + x^3 + x^2 + x + 1).$$

Unfortunately, as is hinted by comparing $P_3(p)/Q_3(p)$ and $P_4(p)/Q_4(p)$, the complexity of the rational functions involved appears to increase as r increases. This has the effect of slowing down the computer calculations.

By factoring values $\zeta(n)$ for larger n from $M_3(h)$, it is observed that this process also appears to increase the complexity of the rational function of the remainder term.

By proceeding in this way we find that

$$M_3(h) = \frac{\zeta^3(2)\zeta^3(3)\zeta^6(7)\zeta^3(8)\zeta^{12}(11)\zeta^{28}(12)\zeta^{105}(16)\zeta^{126}(17)\zeta^{63}(20)\cdots}{\zeta^2(4)\zeta^3(5)\zeta(6)\zeta^{10}(9)\zeta^9(10)\zeta^6(13)\zeta^{63}(14)\zeta^{26}(15)\zeta^{135}(18)\zeta^{360}(19)\cdots}.$$

Note that the ζ-factorization appears to continue indefinitely. This is, in fact, the case. First, we observe that the factors of $\zeta(n)$ are of the form

$$\frac{p^n}{p^n - 1}.$$

Thus, in particular, finite products and quotients of $\zeta(n)$ have factors with numerators and denominators of the form $p^a \prod_i c_i(p)$, where c_i are cyclotomic polynomials. Thus if terms of $M_r(h)$ are not quotients of cyclotomic polynomials in p or powers of p, they cannot be expressed as a finite product of $\zeta(n)$. For instance, for $M_3(h)$, we have

$$1 + s(p,3) = \frac{p^4(p^4 - p^3 + 3p^2 - p + 1)}{(p-1)(p^3-1)(p^4-1)}.$$

The polynomial in p in the numerator is not a product of cyclotomic polynomials, which can be seen by using PARI to identify at least one complex root with norm not equal to 1. Similarly, we find for $M_4(h)$ that $1 + s(p,4)$ has in its numerator a factor $p^6 + 3p^4 + 4p^3 + 3p^2 + 1$, with at least one complex root with norm not equal to 1.

Since in general we cannot hope for a terminating product of terms involving $\zeta(n)$, we discuss how to handle a remainder factor from a ζ-factorization which is an infinite product. In particular, this is how we will bound the values of $M_r(h)$ in the case of $r = 3$ and $r = 4$. We will compute the bounds by multiplying the terms over the primes $p \le p_0$ for some bound p_0, and then estimate the error incurred by truncating the product at p_0. We will call the tail of the product $T(p_0)$, so that

$$T(p_0) = \prod_{p>p_0} \left(1 - \frac{P_r(p)}{Q_r(p)}\right).$$

Using the inequality $\log(1+x) \le x$ which is valid for $x > -1$, we can upper bound $T(p_0)$ by

$$\exp\left(\log \prod_{p>p_0} \left(1 - \frac{P_r(p)}{Q_r(p)}\right)\right) \le \exp\left(-\sum_{p>p_0} \frac{P_r(p)}{Q_r(p)}\right),$$

provided that $P_r(x)/Q_r(x) < 1$ for $x > p_0$. We must then find a lower bound for the sum

$$\sum_{p>p_0} \frac{P_r(p)}{Q_r(p)}.$$

Since we will subsequently want an upper bound for this sum as well, we will treat both bounds concurrently. We will first let c_r^-, c_r^+ denote constants such that

$$\frac{c_r^-}{x^5} \le \frac{P_r(x)}{Q_r(x)} \le \frac{c_r^+}{x^5}$$

for $x > p_0$. Such bounds exist since

$$\frac{P_r(x)}{Q_r(x)} \sim \frac{c}{x^5}$$

for some integer c as $x \to \infty$. Then it remains to bound

$$\begin{aligned}\sum_{p>p_0} \frac{1}{p^5} &= \left[\frac{\pi(t)}{t^5}\right]_{p_0}^{\infty} + 5\int_{p_0}^{\infty} \frac{\pi(t)}{t^6} dt \\ &= -\frac{\pi(p_0)}{p_0^5} + 5\int_{p_0}^{\infty} \frac{\pi(t)}{t^6} dt,\end{aligned}$$

where the first term can be calculated directly.

To bound the integral, we will use the bounds for $\pi(x)$ of Dusart [10],

$$\pi(x) \geq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}\right) \tag{4.19}$$

for $x \geq 32299$ and

$$\pi(x) \leq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\right)$$

for $x \geq 355991$.

We will show in detail only the derivation of the lower bound for

$$\int_{p_0}^{\infty} \frac{\pi(t)}{t^6} dt,$$

as the upper bound proceeds analogously.

By (4.19), we begin by writing

$$\int_{p_0}^{\infty} \frac{\pi(t)}{t^6} dt \geq \int_{p_0}^{\infty}\left(\frac{1}{t^5 \log t}+\frac{1}{t^5 \log^2 t}+\frac{1.8}{t^5 \log^3 t}\right) dt.$$

Since for all real s we have

$$\int_{p_0}^{\infty} \frac{1}{t^5 \log^s t} dt=\frac{1}{4 p_0^4 \log^s p_0}-\frac{s}{4} \int_{p_0}^{\infty} \frac{1}{t^5 \log^{s+1} t} dt, \tag{4.20}$$

we take $s=1,2$, and 3 to arrive at the bound

$$\int_{p_0}^{\infty} \frac{\pi(t)}{t^6} dt \geq \frac{1}{4} \frac{1}{p_0^4 \log p_0}+\frac{3}{16} \frac{1}{p_0^4 \log^2 p_0}+\frac{57}{160} \frac{1}{p_0^4 \log^3 p_0}-\frac{171}{160} \int_{p_0}^{\infty} \frac{1}{t^5 \log^4 t} dt.$$

We next need an upper bound for the integral

$$\int_{p_0}^{\infty} \frac{1}{t^5 \log^4 t} dt,$$

which we can find by noting that

$$\int_{p_0}^{\infty} \frac{1}{t^5 \log^4 t} dt \leq \frac{1}{\log^4 p_0} \int_{p_0}^{\infty} \frac{1}{t^5} dt = \frac{1}{4p_0^4 \log^4 p_0}. \tag{4.21}$$

However, we can do slightly better: If we use the more conservative bound

$$\int_{p_0}^{\infty} \frac{1}{t^5 \log^4 t} dt \leq \frac{1}{\log p_0} \int_{p_0}^{\infty} \frac{1}{t^5 \log^3 t} dt,$$

we may use Equation (4.20) for $s = 3$ to get

$$\int_{p_0}^{\infty} \frac{1}{t^5 \log^4 t} dt \leq \frac{1}{\log p_0} \left(\frac{1}{4p_0^4 \log^3 p_0} - \frac{3}{4} \int_{p_0}^{\infty} \frac{1}{t^5 \log^4 t} dt \right).$$

Solving this inequality for the integral, we arrive at the bound

$$\int_{p_0}^{\infty} \frac{1}{t^5 \log^4 t} dt \leq \frac{1}{4 \log p_0 + 3} \cdot \frac{1}{p_0^4 \log^3 p_0},$$

which is slightly better than Inequality (4.21). Thus,

$$\int_{p_0}^{\infty} \frac{\pi(t)}{t^6} dt \geq \frac{1}{4} \frac{1}{p_0^4 \log p_0} + \frac{3}{16} \frac{1}{p_0^4 \log^2 p_0} + \frac{57}{160} \frac{1}{p_0^4 \log^3 p_0} - \frac{171}{160} \frac{1}{(4 \log p_0 + 3) p_0^4 \log^3 p_0},$$

which simplifies to

$$\int_{p_0}^{\infty} \frac{\pi(t)}{t^6} dt \geq \frac{1}{4} \frac{1}{p_0^4 \log p_0} + \frac{3}{80} \left(\frac{20 \log p_0 + 53}{4 \log p_0 + 3} \right) \frac{1}{p_0^4 \log^2 p_0}.$$

The upper bound calculation begins in the same way, giving

$$\int_{p_0}^{\infty} \frac{\pi(t)}{t^6} dt \leq \frac{1}{4} \frac{1}{p_0^4 \log p_0} + \frac{3}{16} \frac{1}{p_0^4 \log^2 p_0} + \frac{427}{200} \frac{1}{p_0^4 \log^3 p_0} - \frac{1281}{800} \int_{p_0}^{\infty} \frac{1}{t^5 \log^4 t} dt.$$

At this point the upper bound calculation uses, instead of (4.20), the inequality

$$\begin{aligned} \int_{p_0}^{\infty} \frac{1}{t^5 \log^4 t} dt &= \frac{1}{4p_0^4 \log^4 p_0} - \int_{p_0}^{\infty} \frac{1}{t^5 \log^5 t} dt \\ &\geq \frac{1}{4p_0^4 \log^4 p_0} - \int_{p_0}^{\infty} \frac{1}{t^5 \log^4 t} dt. \end{aligned}$$

Now solving for our desired bound, we have

$$\int_{p_0}^{\infty} \frac{1}{t^5 \log^4 t} dt \geq \frac{1}{8p_0^4 \log^4 p_0}.$$

Thus

$$\int_{p_0}^{\infty} \frac{\pi(t)}{t^6} dt \leq \frac{1}{4} \frac{1}{p_0^4 \log p_0} + \frac{3}{16} \frac{1}{p_0^4 \log^2 p_0} + \frac{427}{200} \frac{1}{p_0^4 \log^3 p_0} - \frac{1281}{6400} \frac{1}{p_0^4 \log^4 p_0}. \quad (4.22)$$

Collecting the appropriate bounds for an upper bound for $T(p_0)$, which we will denote $T^+(p_0)$, we have

$$T^+(p_0) \quad = \quad \exp\left(c_r^- \left(\frac{\pi(p_0)}{p_0^5} - \frac{5}{4} \cdot \frac{1}{p_0^4 \log p_0} - \frac{15}{80} \left(\frac{20 \log p_0 + 53}{4 \log p_0 + 3} \right) \frac{1}{p_0^4 \log^2 p_0} \right) \right).$$

Next we find a lower bound $T^-(p_0)$ for $T(p_0)$. To do this we must bound below the product

$$\prod_{p>p_0}\left(1-\frac{P_r(p)}{Q_r(p)}\right)=\exp\left(\sum_{p>p_0}\log\left(1-\frac{P_r(p)}{Q_r(p)}\right)\right).$$

Using the power series for $-\log(1-x)$, we have the upper bound

$$\begin{aligned}-\log(1-x)&=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\\&\le x+\frac{x^2}{2}\left(\frac{1}{1-x}\right)\end{aligned}$$

for $0\le x<1$. Thus if $P_r(x)/Q_r(x)$ is decreasing and is in the interval $[0,1)$ for $x>p_0$, then we have

$$\exp\left(\sum_{p>p_0}\log\left(1-\frac{P_r(p)}{Q_r(p)}\right)\right)\ge\exp\left(-\sum_{p>p_0}\frac{P_r(p)}{Q_r(p)}-\frac{1}{2\left(1-\frac{P_r(p_0)}{Q_r(p_0)}\right)}\sum_{p>p_0}\frac{P_r^2(p)}{Q_r^2(p)}\right).$$

Again if $P_r(x)/Q_r(x)$ is decreasing, then so is $P^2(x)/Q^2(x)$ and we have the inequality

$$\sum_{p>p_0}\frac{P_r^2(p)}{Q_r^2(p)}\le\int_{p_0}^{\infty}\frac{P_r^2(t)}{Q_r^2(t)}dt.$$

Using this along with our bound (4.22), we can let $T^-(p_0)$ be

$$\exp\left(c_r^+\left(\frac{\pi(p_0)}{p_0^5}-\frac{5}{4}\frac{1}{p_0^4\log p_0}-\frac{15}{16}\frac{1}{p_0^4\log^2 p_0}-\frac{427}{40}\frac{1}{p_0^4\log^3 p_0}+\frac{1281}{1280}\frac{1}{p_0^4\log^4 p_0}\right)\right.$$
$$\left.-\frac{1}{2\left(1-\frac{P_r(p_0)}{Q_r(p_0)}\right)}\int_{p_0}^{\infty}\frac{P_r^2(t)}{Q_r^2(t)}dt\right).$$

The conditions on $P_r(x)/Q_r(x)$ are checked for each of the cases $r=3$ and $r=4$.

It is found for both $r = 3$ and $r = 4$ that $P_r(x)/Q_r(x)$ is less than 1 for all $x > 1$ and decreases for $x > 0$. Thus $\lim_{x\to\infty} P_r(x)/Q_r(x) = 0$, and we have $0 \le P_r(x)/Q_r(x) < 1$ for $x > 1$. We find that for $p_0 = 10^6$, we can take $c_3^- = 3$, $c_3^+ = 3.000001$, $c_4^- = 11$, and $c_4^+ = 11.000011$. We calculate $T^+(p_0)$ and $T^-(p_0)$ for each r to find that

$$M_3(h_{500}) = 1.00082088048923772983550566523 \pm 5.08 \times 10^{-27}$$

and

$$M_4(h_{500}) = 1.00109523033158618992636631361 \pm 1.87 \times 10^{-26}.$$

The error in these calculations is due to using $c_r^{\pm}/p^5$ as upper and lower bound approximations to $P_r(x)/Q_r(x)$. If we needed higher precision we could improve the bounds by proceeding as follows: By polynomial division we find, say for $r = 3$, that

$$\frac{P_3(x)}{Q_3(x)} = \frac{3}{p^5} + \frac{1}{p^6} - \frac{6}{p^7} - \frac{3}{p^8} - \cdots .$$

The reciprocal sums of prime powers can then be calculated as in [17], where Möbius inversion is used to write the sum in terms of the Riemann ζ-function as

$$\sum_p \frac{1}{p^s} = \sum_{i=1}^{\infty} \frac{\mu(k)}{k} \log \zeta(ks).$$

As we have seen, computing integral values of ζ is fast, so we can calculate successive sums of primes over $3/p^5$, $1/p^6$, and so on to high precision.

To summarize, we find the following exact values for the moments of h_{500} up to the decimal digits shown. We have also shown in parentheses the original Deléglise upper bounds for the moments M_1, M_2, and M_4 for comparison.

$$M_1(h_{500}) = 1.0002732659660536234324303108 7 \pm 5.6 \times 10^{-30}, \quad (1.000273298199\ldots)$$

$$M_2(h_{500}) = 1.000546892582885088415522752 72 \pm 2.7 \times 10^{-30}, \quad (1.000546957066\ldots)$$

$$M_3(h_{500}) = 1.000820880489237729835505665 23 \pm 5.08 \times 10^{-27},$$

$$M_4(h_{500}) = 1.001095230325222575502928409 862 \pm 1.88 \times 10^{-26}. \quad (1.001095359363\ldots)$$

To determine how these new moments affect the value of the Deléglise bound, we make the following modifications to the Deléglise code. Recall that Deléglise's program uses only rth moments with r equalling powers of 2. We replace the Deléglise values for the upper bounds of the moments for $r = 1, 2$, and 4 with our new values. We also include the value of the moment for $r = 3$. Keeping the parameters $y = 500$ and $z = 10^{14}$ fixed along with the Deléglise bounds for moments with $r > 4$, we find the new upper bound for the density of abundant numbers

$$0.24796597989\ldots \quad (0.24796600460\ldots)$$

(where the previous upper bound is shown in parentheses for comparison). This is an improvement of about 2.47×10^{-8} from the Deléglise upper bound. Thus this method does not, by itself, justify the effort required to implement it. In fact, we will be making good use of these high-precision moment values in the next subsection.

The reduced moment method

In this section we modify the Behrend moment method from Proposition 2.16 with the goal of improving on the upper bound. The Behrend moment method can be viewed as starting with the polynomial $P(x) = x^r$, and using Proposition 2.14 on the arithmetic function $P(h_y)$. By choosing a different polynomial, we could hope for an improved upper bound. One such choice yields the following proposition.

Proposition 4.8. *For each integer $r \geq 1$ and $\alpha > 1$, we have*

$$\mathbf{d}\,\mathscr{A}_{y,\alpha} \leq F(y)\frac{M_r(h_y - 1)}{(\alpha - 1)^r},$$

where

$$M_r(h_y - 1) = \sum_{i=0}^{r}(-1)^i\binom{r}{i}M_i(h_y).$$

Proof. We repeat the argument for Proposition 2.16 using Proposition 2.14 with arithmetic function $P(h_y)$, where $P(x) = (x-1)^r+1$, and $\alpha_0 = 1$. Note that $P(h_y(n)) \geq 1$ for all n. Also observe that the mean of $P(h_y)$ exists and is a linear combination of the ith moments of h_y for $i = 0, \ldots, r$. □

We will call the moments $M_r(h_y - 1)$ the *reduced rth moments of h_y*. Note that the computation of $M_r(h_y - 1)$ for $r > 1$ involves negative terms. Thus the Deléglise upper bounds for $M_r(h_y)$ cannot be used since these do not in general have sufficient precision. Instead we use the ζ-factor method to determine the first few moments to many decimal places, and then use these values to compute the reduced moments.

Based on our calculations of $M_r(h_{500})$ for $r = 1, \ldots, 4$, we find

$$M_1(h_{500} - 1) = 2.7326596605362343243031087 \times 10^{-4} \pm 5.6 \times 10^{-30},$$

$$M_2(h_{500}-1) = 3.6065077784155066213098 \times 10^{-7} \pm 1.39 \times 10^{-29},$$

$$M_3(h_{500}-1) = 6.3874333488622833968 \times 10^{-10} \pm 5.11 \times 10^{-27},$$

$$M_4(h_{500}-1) = 1.37087245067671054 \times 10^{-12} \pm 3.92 \times 10^{-26}.$$

Note that, compared to the Behrend moment method, the reduced moment method has the disadvantage that all moments $M_s(h_y)$ for $s \le r$ must be known to high precision to calculate $M_r(h_y - 1)$. Thus, in practice, we may calculate $M_s(h_y - 1)$ up to a certain point, as we have done up to $r = 4$.

Recall that the Deléglise bound does not do better than the trivial bound for values near $\alpha = 1$. This is true for the reduced moment method as well. We can see this by writing

$$F(y)\frac{M_r(h_y-1)}{(\alpha-1)^r} < F(y) \qquad \iff \qquad \alpha > \sqrt[r]{M_r(h_y-1)} + 1.$$

Thus for sufficiently small α, the asymptotic method outperforms the reduced moment method.

Using the reduced moment method up to $r = 4$, we find for $y = 500$, $z = 10^{14}$ that

$$\mathbf{d}\,\mathscr{A} \le 0.24794525016,$$

which improves the Deléglise upper bound by about 2.07×10^{-5}.

To conclude this subsection we provide a table of upper bounds for $\mathbf{d}\,\mathscr{A}_{500}(\alpha)$ for various values of α. We will compare Deléglise's upper bounds with the asymptotic bound and reduced moment bound. Note that Deléglise's bounds are better for large α, while the asymptotic bound is better for small α. The reduced moment bound performs somewhere in between. However, it should be noted that Deléglise used very high moments for his method, while we have only used moments up to $r = 4$ for the reduced moments method.

α	Deléglise	asymptotic	reduced
1.00001	0.08961	0.04578	0.08961
1.0001	0.08961	0.03634	0.08961
1.001	0.02450	0.02195	0.02449
1.0015	0.01604	0.01859	0.01437
1.002	0.009414	DNE	0.007155
1.005	4.256×10^{-5}	DNE	1.966×10^{-4}
1.01	1.640×10^{-9}	DNE	1.229×10^{-5}

Remark 4.9. Since Deléglise's bounds at high moments are better than the reduced moments up to $r = 4$ for large α, we can improve the density upper bound by either calculating reduced moments up to a level comparable to that of Deléglise, or simply use the Deléglise bounds for large α. In what follows, this is not done, but rather only the asymptotic method and the reduced moment method are used. The Deléglise bounds will be incorporated along with the two new methods in future work.

4.2.2 Piggybacking onto the large primes method

We return our attention to the Deléglise method of bounding the density of non-deficient numbers in Section 3.1. Recall that there are two approximations made: bounding the densities of the sets $\mathscr{M}_y(\{n\}) \cap \mathscr{A}$ for y-smooth $n \leq z$, and using the density of $\mathscr{M}_y(\{n\})$ for y-smooth $n > z$ in place of only the nondeficient members of this set. Note in the first idea that if $n \leq z$ is nondeficient, then the density of the set $\mathscr{M}_y(\{n\}) \cap \mathscr{A}$ is in fact $\mathscr{M}_y(\{n\})$, and this density is accounted for in the lower bound. Thus we can consider the part of the upper bound sum

$$\sum_{\substack{n \le z \\ P(n) \le y \\ h(n) < 2}} \frac{\tilde{A}_{y,2/h(n)}}{n}$$

as "piggybacking" onto the lower bound sum

$$\sum_{\substack{n \le z \\ P(n) \le y \\ h(n) \ge 2}} \frac{F(y)}{n}. \tag{4.23}$$

In this section we consider how we can analogously piggyback an upper bound density onto each of the large primes method lower bounds. We will first examine the nondeficient numbers not yet considered in the single large primes method to determine an upper bound for the density of these numbers. We will afterwards look at how this is adapted to include also the double large primes method.

Recall from Subsection 4.1.3 that the single large primes lower bound method computes the density of the sets $\mathscr{M}_{p-1}(\{np\})$ for y-smooth deficient $n \le z$ such that $p \in (y, h(n)/(2-h(n))]$. These sets are considered in conjunction with the sets $\mathscr{M}_y(\{n\})$ for y-smooth nondeficient $n \le z$ to determine a lower bound for the density of nondeficient numbers. As before, let nm be a number such that n is y-smooth and $(m, P(y)) = 1$. Then either $n \le z$ or $n > z$. For these numbers $n > z$, we retain Deléglise's expression

$$1 - F(y) \sum_{\substack{n \le z \\ P(n) \le y}} \frac{1}{n} \tag{4.24}$$

for an upper bound for the density of such nondeficient nm. If $n \le z$, then the remaining nondeficient numbers nm are such that $h(n) < 2$. The single large primes method handles the case

$$2 - \frac{2}{y+1} < h(n) < 2 \tag{4.25}$$

and

$$y < p \leq \frac{h(n)}{2 - h(n)}.$$

Suppose $h(n)$ satisfies the bounds 4.25 but $p(m) > h(n)/(2 - h(n))$. Nondeficient numbers in this case have not been included in the single large primes method. Thus we must consider the density of the set

$$\mathscr{M}_{\frac{h(n)}{2-h(n)}}(\{n\}) \cap \mathscr{A}.$$

But we know that

$$\mathbf{d}\,\mathscr{M}_{\frac{h(n)}{2-h(n)}}(\{n\}) \cap \mathscr{A} = \frac{1}{n}\,\mathbf{d}\,\mathscr{A}_{\frac{h(n)}{2-h(n)},\frac{2}{h(n)}}.$$

Summing over the numbers n in this case, we arrive at the density expression

$$\sum_{\substack{n \leq z \\ P(n) \leq y \\ 2 - \frac{2}{y+1} < h(n) < 2}} \frac{1}{n}\,\mathbf{d}\,\mathscr{A}_{\frac{h(n)}{2-h(n)},\frac{2}{h(n)}}. \tag{4.26}$$

We also have the case $h(n) \leq 2 - 2/(y+1)$ to consider. In this case, we may proceed as did Deléglise and use the bound

$$\sum_{\substack{n \leq z \\ P(n) \leq y \\ h(n) \leq 2 - \frac{2}{y+1}}} \frac{1}{n}\,\mathbf{d}\,\mathscr{A}_{y,\frac{2}{h(n)}}. \tag{4.27}$$

Combining the density bounds (4.23), (4.3), (4.26), (4.27), and (4.24), we arrive at the upper bound expression

$$\mathbf{d}\mathscr{A} \le \sum_{\substack{n\le z\\ P(n)\le y\\ h(n)\ge 2}} \frac{F(y)}{n} + \sum_{\substack{n\le z\\ P(n)\le y\\ 2-\frac{2}{y+1}<h(n)<2}} \frac{1}{n}\left(F(y) - F\left(\frac{h(n)}{2-h(n)}\right)\right)$$
$$+ \sum_{\substack{n\le z\\ P(n)\le y\\ 2-\frac{2}{y+1}<h(n)<2}} \frac{1}{n}\mathbf{d}\mathscr{A}_{\frac{h(n)}{2-h(n)},\frac{2}{h(n)}} + \sum_{\substack{n\le z\\ P(n)\le y\\ h(n)\le 2-\frac{2}{y+1}}} \frac{1}{n}\mathbf{d}\mathscr{A}_{y,\frac{2}{h(n)}}$$
$$+ 1 - F(y)\sum_{\substack{n\le z\\ P(n)\le y}} \frac{1}{n}. \quad (4.28)$$

We will call this the *single large primes upper bound method.*

Remark 4.10. It may be possible to find upper bounds corresponding to the small or medium primes lower bound methods. However, the attempts so far have not yielded any improvements.

We next consider piggybacking onto the two large primes lower bound method. The sets of nondeficient numbers which are not considered by the Deléglise, single large primes, and two large primes methods fall into four categories. First, we have the case where for y-smooth n we have $n > z$, which we will handle as above with the sum (4.24).

The remaining cases involve y-smooth $n \le z$. Second, we have the case $h(np_1p_2) < 2$, where p_1 and p_2 are the first two primes greater than y. Such n are too deficient for the two large primes method to apply. We will bound the density of such multiples of n by

$$\frac{\mathbf{d}\mathscr{A}_{y,\frac{2}{h(n)}}}{n} \quad (4.29)$$

as in the Deléglise method.

Third, we have $h(n) \in (b_2, 2)$ and $p > a_2$, where b_2 and a_2 are defined in Subsection 4.1.3. This is the case where there are no primes $q > p$ such that $h(npq) \geq 2$. The nondeficient numbers corresponding to this case are the multiples mn where $(m, \Pi(a_2)) = 1$ and $h(mn) \geq 2$, which have density

$$\frac{\mathbf{d}\,\mathscr{A}_{a_2, \frac{2}{h(n)}}}{n}. \tag{4.30}$$

Finally, we have $h(n) \in (b_2, 2)$, $a_1 < p \leq a_2$, where b_1 and a_1 are also defined in Subsection 4.1.3. This is the case where there are primes $q > p$ such that $h(npq) \geq 2$. However, $h(npq') < 2$ for sufficiently large primes q', namely when $q' > h(np)/(2 - h(np))$. Repeating our argument for the single large primes upper bound method, we find that the corresponding density is

$$\frac{\mathbf{d}\,\mathscr{A}_{\frac{h(np)}{2-h(np)}, \frac{2}{h(np)}}}{np}.$$

In fact, we must account for powers of primes p as well. We do this by using the upper bound

$$\sum_{i=1}^{\infty} \frac{\mathbf{d}\,\mathscr{A}_{\frac{h(np^i)}{2-h(np^i)}, \frac{2}{h(np^i)}}}{np^i} \leq \frac{\mathbf{d}\,\mathscr{A}_{\frac{h(np)}{2-h(np)}, \frac{2}{h(n)h(p^\infty)}}}{n(p-1)}, \tag{4.31}$$

due to the monotonicity of $\mathbf{d}\,\mathscr{A}_{y,\alpha}$ in each of y and α. Here $h(p^\infty)$ is shorthand for $\lim_{i\to\infty} h(p^i) = p/(p-1)$. The infinite sum can be seen to represent the density of the corresponding infinite union as before in Subsection 4.1.3 by convergence of the geometric series.

Thus piggybacking onto the two large primes lower bound method we have the combined density from (4.24), (4.29), (4.30), and (4.31), namely

$$\sum_{\substack{n\le z\\ P(n)\le y\\ h(n)<2/h(p_1p_2)}} \frac{\mathbf{d}\,\mathscr{A}_{y,\frac{2}{h(n)}}}{n} + \sum_{\substack{n\le z\\ P(n)\le y\\ h(n)\in[2/h(p_1p_2),2)}} \frac{1}{n}\left(\mathbf{d}\,\mathscr{A}_{a_2,\frac{2}{h(n)}} + \sum_{\max\{y,a_1\}<p\le a_2} \frac{\mathbf{d}\,\mathscr{A}_{\frac{h(np)}{2-h(np)},\frac{2}{h(n)h(p^\infty)}}}{p-1}\right) + 1 - F(y)\sum_{\substack{n\le z\\ P(n)\le y}} \frac{1}{n}. \quad (4.32)$$

Thus we have found what we call the *two large primes upper bound* by adding (4.10) and (4.32).

Proposition 4.11. *The density of abundant numbers is bounded above by*

$$\mathbf{d}\,\mathscr{A} \le \sum_{\substack{n\le z\\ P(n)\le y\\ b_2<h(n)<2}} \frac{1}{n}\left(F(y) - F(a_2) + \mathbf{d}\,\mathscr{A}_{a_2,\frac{2}{h(n)}}\right) - \sum_{\substack{n\le z\\ P(n)\le y\\ b_2<h(n)<2}} \frac{1}{n} \sum_{\max\{y,a_1\}<p\le a_2} \left(\frac{F\left(\frac{h(np)}{2-h(np)}\right) - \mathbf{d}\,\mathscr{A}_{\frac{h(np)}{2-h(np)},\frac{2}{h(n)h(p^\infty)}}}{p-1}\right) + \sum_{\substack{n\le z\\ P(n)\le y\\ h(n)<2/h(p_1p_2)}} \frac{\mathbf{d}\,\mathscr{A}_{y,\frac{2}{h(n)}}}{n} + 1 - F(y)\sum_{\substack{n\le z\\ P(n)\le y\\ h(n)<2}} \frac{1}{n},$$

where $a_i = \sqrt[i]{h(n)}/(\sqrt[i]{2} - \sqrt[i]{h(n)})$ *and* $b_i = 2(1-1/(y+1))^i$.

Computational issues

We now address several computational issues that arise due to the size of the primes required, which may exceed our bound $p_{\max}$.

To begin, we will address the inner sum in the density bound of Proposition 4.11. When $p_{\max} < a_2$, we bound the sum

$$\sum_{\max\{p_{\max}, a_1\} < p \leq a_2} \left(\frac{F\left(\frac{h(np)}{2-h(np)}\right) - \mathbf{d}\,\mathscr{A}_{\frac{h(np)}{2-h(np)}, \frac{2}{h(n)h(p^\infty)}}}{p-1} \right)$$

analogously as in the corresponding lower bound method. Namely, we replace the expression $F(h(np)/(2-h(np))$ by its minimum possible value $F(a_2)$, and note that $\mathbf{d}\,\mathscr{A}_{y,\alpha}$ increases when y decreases, and also increases when α decreases. Thus we replace the expressions

$$\frac{h(np)}{2-h(np)} \qquad \text{and} \qquad \frac{2}{h(n)h(p^\infty)}$$

by their smallest possible values for $p \in (\max\{p_{\max}, a_1\}, a_2]$. These are

$$\frac{h(n)\left(1+\frac{1}{a_2}\right)}{2-h(n)\left(1+\frac{1}{a_2}\right)} \qquad \text{and} \qquad \frac{2}{h(n)\left(1+\frac{1}{p_{\max}}\right)},$$

respectively. Then it remains to bound

$$\sum_{p \in (p_{\max}, a_2]} \frac{1}{p-1}.$$

We do this by taking
$$\frac{1}{p-1} \leq \frac{1}{p}\left(1+\frac{1}{p_{\max}-1}\right),$$
and then the sum
$$\sum_{p\in(p_{\max},a_2]} \frac{1}{p}$$
is bounded by using Dusart's bounds (4.11) on the sum of reciprocal primes.

Computing reduced moments $M_r(h_y-1)$

In order to implement the foregoing methods into a computer program, it would be useful to have available upper bounds for $\mathbf{d}\,\mathscr{A}_{y,\alpha}$ for large values of y. We next discuss two ways in which these can be found. The first discusses the strategy for y that are small enough so that all primes up to some $p_{\max}$ can be computed, in which case we will be using the reduced moment method. This means that we need upper bounds on the reduced moments $M_r(h_y-1)$ for $y \leq p_{\max}$. The second method addresses the case where $y > p_{\max}$, in which case we will simply use an estimate for the Deléglise first moment upper bound expression.

Calculating $M_r(h_y-1)$ for $y \leq p_{\max}$. One way to find the values $M_r(h_y-1)$ is to let PARI do all the calculations beforehand and store the values in a data file for the program to access. This approach works, but it would be preferable not to have to store these values but rather to compute them at runtime, since the size of a file storing the rth moments up to $r_{\max}$ would be on the order of $10\cdot r_{max}\cdot\pi(p_{\max})$ bytes, assuming a long double precision float takes 10 bytes. This is about 120 megabytes for our choice of $r_{max}=4$ and $p_{\max}=5\times10^7$. However, we must be careful when implementing a computation for $M_r(h_y-1)$. We will illustrate the issue first for the case $M_r(h_y)$.

A problem which arises when storing values of $M_r(h_y)$ on a computer is that for large y, the values of $M_r(h_y)$ behave like $1+\epsilon$ for some small $\epsilon > 0$. For instance, we have seen that the first few moments of h_{500} are of the form $1.000*$. In this case we would prefer to store the value $M_r(h_y) - 1$, as this would allow us to keep higher precision. In fact, this is precisely how Deléglise stores the moments in his program.

This point is underscored when we try to compute values of $M_r(h_y - 1)$. Recall that the fourth reduced moment has a value of about

$$M_4(h_{500} - 1) = 1.371 \times 10^{-12}.$$

If we were to attempt to calculate this moment directly from the moments $M_r(h_{500})$, $r = 1, \ldots, 4$, then even with long double precision which allows 19 digits to be stored, we would end up with only 7 significant digits for $M_4(h_y - 1)$. This would only get worse for larger values of r.

We first prove the stated behavior of $M_r(y)$ and then show how the issue of calculating $M_r(h_y - 1)$ may be solved.

The behavior of $M_r(h_y)$. In order to understand the behavior of $M_r(h_y)$ as y increases, we first estimate $\rho_y(p^k, r) := h_y(p^k)^r - h_y(p^{k-1})^r$. By the identity $(a^r - b^r) = (a-b)(a^{r-1} + \cdots + b^{r-1})$ and the observation that $h_y(p^k) = h_y(p^{k-1}) + \frac{1}{p^k}$ when $p > y$, we have

$$\begin{aligned}\rho_y(p^k, r) &= h_y(p^k)^r - h_y(p^{k-1})^r \\ &= (h_y(p^k) - h_y(p^{k-1}))(h_y(p^k)^{r-1} + \cdots + h_y(p^{k-1})^{r-1})\end{aligned}$$

$$\leq \frac{1}{p^k} \cdot r h_y(p^k)^{r-1}$$
$$< \frac{r}{p^k} \cdot \left(\frac{p}{p-1}\right)^{r-1}$$
$$\leq \frac{r}{p^k} \cdot \left(1 + \frac{1}{y-1}\right)^{r-1}.$$

Thus $s(p,r) = \frac{\rho(p)}{p} + \frac{\rho(p^2)}{p^2} + \cdots$, which we defined in subsection 4.2.1, is bounded by

$$r\left(1 + \frac{1}{y-1}\right)^{r-1} \sum_{k=1}^{\infty} \frac{1}{p^{2k}} = \frac{C_{r,y}}{p^2-1},$$

where $C_{r,y} := r(1 + 1/(y-1))^{r-1}$ only depends on r and y. Note that as $y \to \infty$, $C_{r,y} \to r$, and also that $\sum \frac{1}{p^2-1}$ converges. Thus,

$$M_r(h_y) \leq \prod_{p>y} \left(1 + \frac{C_{r,y}}{p^2-1}\right) \leq \exp\left(C_{r,y} \sum_{p>y} \frac{1}{p^2-1}\right),$$

and writing $L_r(h_y) := M_r(h_y) - 1$, we have $L_r(h_y) \to 0$ as $y \to \infty$. In particular, we find that

$$\begin{aligned} L_r(h_y) &= \left(r + \frac{r(r-1)}{y} + O\left(\frac{r(r-1)}{y^2}\right)\right)\left(\frac{1}{y\log y} + O\left(\frac{1}{y\log^2 y}\right)\right) \\ &= \frac{r}{y \log y} + O_r\left(\frac{1}{y\log^2 y}\right). \end{aligned} \tag{4.33}$$

Thus from a computational standpoint, we see that it is preferable to work with the values of $L_r(h_y)$ rather than $M_r(h_y)$ since in general $M_r(h_y)$ consists of a 1 followed by a string of 0's and then the digits of $L_r(h_y)$.

If we are to work with $L_r(h_y)$, we need an expression for finding $M_r(h_y - 1)$ in terms of $L_r(h_y)$. This is easily found by the following calculation:

$$\begin{aligned} M_r(h_y-1) &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}M_i(h_y) \\ &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}(L_i(h_y)+1) \\ &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}L_i(h_y)+0^r \\ &= \sum_{i=1}^{r}\binom{r}{i}(-1)^{r-i}L_i(h_y)+0^r, \end{aligned}$$

since $\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i} = (1-1)^r = 0^r$, and for the purposes of inversion we define $L_0(h_y) = 0$, and adopt the convention that $0^0 = 1$.

Computing $L_r(h_y)$. In order to compute the value $L_r(h_y)$ for a particular y, we can begin with the value of $L_r(h_{y_1})$ for a large value y_1, $y_1 > y$. Then by using a relation between $L_r(h_{p_i})$ and $L_r(h_{p_{i-1}})$, we can iteratively calculate $L_r(h_y)$ for any value $y < y_1$. We will now find such an iterative relation.

We first observe that

$$\begin{aligned} 1+L_r(h_{p_{i-1}}) &= M_r(h_{p_{i-1}}) \\ &= M_r(h_{p_i})(1+s(p_i,r)) \\ &= (1+L_r(h_{p_i}))(1+s(p_i,r)) \\ &= 1+L_r(h_{p_i})+s(p_i,r)+L_r(h_{p_i})s(p_i,r). \end{aligned}$$

Thus

$$L_r(h_{p_{i-1}}) = L_r(h_{p_i})+s(p_i,r)+L_r(h_{p_i})s(p_i,r), \tag{4.34}$$

so that the calculation of $L_r(h_y)$ no longer depends on $M_r(h_y)$.

A further refinement. As we have seen, computing $L_r(h_y)$ directly allows us to use expressions of size approximately $r/y \log y$. In fact, for higher moments we can do better. For instance, for $r = 2$, $M_2(h_y - 1) = L_2(h_y) - 2L_1(h_y)$, which is

$$O\left(\frac{1}{y \log^2 y}\right)$$

by (4.33).

As a concrete example we have calculated for $y = 5 \times 10^7$ that $L_1(h_y) \approx 1.07 \times 10^{-9}$ and $L_2(h_y) \approx 2.14 \times 10^{-9}$, while

$$L_2(h_y) - 2L_1(h_y) \approx 1.212536 \times 10^{-17}.$$

To take advantage of this situation we simply iterate the strategy used to solve the initial moment calculation problem by again letting the contributions from the main terms of $L_r(h_y)$ cancel while keeping the remainder terms. Just as we have defined $L_r(h_y)$ as the secondary term in $M_r(h_y)$, we will define $K_r(h_y) = M_r(h_y - 1)$ where we view $K_r(h_y)$ as a smaller order term of $L_r(h_y)$, so that

$$K_r(h_y) := \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i} L_i(h_y) \tag{4.35}$$

for integers $r > 0$, and $K_0(h_y) = 0$. (Note that for $r = 1$, $K_1(h_y)$ coincides with

$L_1(h_y)$.) By (4.34) and (4.35), we have

$$\begin{aligned}K_r(h_{p_{i-1}}) &= \sum_{j=1}^{r}\binom{r}{j}(-1)^{r-j}L_j(h_{p_{i-1}})\\ &= \sum_{j=1}^{r}\binom{r}{j}(-1)^{r-j}(L_j(h_{p_i}) + s(p_i,j) + L_j(h_{p_i})s(p_i,j))\\ &= K_r(h_{p_i}) + \sum_{j=1}^{r}\binom{r}{j}(-1)^{r-j}s(p_i,j) + \sum_{j=1}^{r}\binom{r}{j}(-1)^{r-j}L_j(h_{p_i})s(p_i,j).\end{aligned}$$

To remove the terms with $L_j(h_{p_i})$ in the final sum above we note that by inversion (see, for instance, [20, p. 192–3]) we have

$$K_r(h_{p_i}) = \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}L_j(h_{p_i}) \iff L_r(h_{p_i}) = \sum_{j=0}^{r}\binom{r}{j}K_j(h_{p_i}),$$

where we have used the observation that $M_0(h_y) = 1$ and so $L_0(h_y) = 0$. Thus

$$\begin{aligned}\sum_{j=1}^{r}\binom{r}{j}(-1)^{r-j}L_j(h_{p_i})s(p_i,j) &= \sum_{j=1}^{r}\binom{r}{j}(-1)^{r-j}s(p_i,j)\sum_{k=1}^{j}\binom{j}{k}K_k(h_{p_i})\\ &= \sum_{k=1}^{r}\binom{r}{k}K_k(h_{p_i})\sum_{j=1}^{r}\binom{r-k}{j-k}(-1)^{r-j}s(p_i,j),\end{aligned}$$

where we have used that $K_0(h_{p_i}) = 0$ and the convention that $\binom{n}{k} = 0$ when $k > n$. We conclude that

$$\begin{aligned}&K_r(h_{p_{i-1}}) - K_r(h_{p_i}) =\\ &\qquad \sum_{j=1}^{r}\binom{r}{j}(-1)^{r-j}s(p_i,j) + \sum_{k=1}^{r}\binom{r}{k}K_k(h_{p_i})\sum_{j=1}^{r}\binom{r-k}{j-k}(-1)^{r-j}s(p_i,j).\end{aligned}$$

By using this equation we are able to calculate values for $K_r(h_y), r = 1,\ldots,4$, beginning from $y = 5 \times 10^7$ down to $y = 0$ while preserving high precision. As an example, for our fourth moment calculation we begin with the value of $K_4(h_y)$, $y = 5 \times 10^7$, calculated by PARI using the ζ-factor method,

$$3.29594356911638569265174956894439063553 \times 10^{-33}.$$

Calculating all values of $K_4(h_y)$ down to $y = 0$, we arrive at the value

$$1.58798408739662063,$$

which agrees with the value found using the ζ-factor method by PARI to the given number of digits.

Bounding d$\mathscr{A}_{y,\alpha}$ for $y > p_{\max}$. In the case that $y > p_{\max}$, we will use the first moment Deléglise upper bound method. Thus, we need an upper bound approximation for $M(h_y)$, as well as an upper bound approximation for $F(y)$. In addition, the large primes upper bound method requires bounding $-F(y)$ for large y, so we will also need a lower bound for $F(y)$.

In fact, we can approximate these values using partial summation and Dusart's bounds for $\pi(y)$ and $F(y)$, (4.19), (4.20), and (4.4).

It remains to bound above $M(h_y)$. Since

$$M(h_y) = \prod_{p>y}\left(1 - \frac{1}{p^2}\right)^{-1} = \exp\left(-\sum_{p>y}\log\left(1 - \frac{1}{p^2}\right)\right) = \exp\sum_{p>y}\sum_{k=1}^{\infty}\frac{1}{kp^{2k}},$$

we seek an upper bound for the double sum in the exponent. We can bound the inner sum from above by writing

$$\sum_{k=1}^{\infty} \frac{1}{kp^{2k}} \leq \sum_{k=1}^{\infty} \frac{1}{p^{2k}} = \frac{1}{p^2-1}.$$

Then by partial summation and Dusart's bounds [10]

$$\frac{y}{\log y}\left(1+\frac{1}{\log x}\right) \underset{x\geq 599}{\leq} \pi(y) \underset{y>1}{\leq} \frac{y}{\log y}\left(1+\frac{1.2762}{\log x}\right)$$

for $\pi(y)$, which are valid for $y > 599$, we have

$$\begin{aligned}\sum_{p>y} \frac{1}{p^2-1} &= \left[\frac{\pi(t)}{t^2-1}\right]_y^{\infty} + 2\int_y^{\infty} \frac{\pi(t)}{t^3} dt \\ &\leq -\frac{y}{(y^2-1)\log y}\left(1+\frac{1}{\log y}\right) + 2\int_y^{\infty} \frac{1}{t^2 \log t}\left(1+\frac{1.2762}{\log t}\right) dt.\end{aligned}$$

Now by integration by parts we have that

$$\int_y^{\infty} \frac{1}{t^2 \log t} = \left[-\frac{1}{t\log t}\right]_y^{\infty} - \int_y^{\infty} \frac{dt}{t^2 \log^2 t} = \frac{1}{y \log y} - \int_y^{\infty} \frac{dt}{t^2 \log^2 t}.$$

Thus,

$$\begin{aligned}\sum_{p>y} \frac{1}{p^2-1} &\leq -\frac{1}{y\log y}\left(1+\frac{1}{\log y}\right) + 2\left(\frac{1}{y\log y} - \int_y^{\infty} \frac{dt}{t^2\log^2 t}\right) + 2\int_y^{\infty} \frac{1.2762}{t^2 \log^2 t} dt \\ &= \frac{1}{y\log y} - \frac{1}{y\log^2 y} + \int_y^{\infty} \frac{0.5524}{t^2\log^2 t} dt \\ &= \frac{1}{y\log y} - \frac{1}{y\log^2 y} + \frac{1}{\log^2 y}\int_y^{\infty} \frac{0.5524}{t^2} dt \\ &= \frac{1}{y\log y} - \frac{0.4476}{y\log^2 y}.\end{aligned}$$

Thus,

$$M(h_y) \leq \exp\left(\frac{1}{y\log y} - \frac{0.4476}{y\log^2 y}\right).$$

Note that since the argument in the exponential is small, the value after taking the exponential is about 1 greater than the argument. In fact, we are actually interested in the value $M(h_y) - 1$. We can thus attempt to simplify this expression by removing the exponential. Since

$$e^x - 1 \leq x + x^2$$

for $x \leq 3/4$, and

$$\frac{1}{y\log y} \leq \frac{4}{3}$$

for $y \geq 2$, we end up with the following bound.

Proposition 4.12. *When $y > 599$, we have the explicit upper bound*

$$M(h_y) - 1 \leq \frac{1}{y\log y} - \frac{0.4476}{y\log^2 y} + \left(\frac{1}{y\log y} - \frac{0.4476}{y\log^2 y}\right)^2.$$

Large prime upper bound results. With upper bounds for $\mathbf{d}\mathscr{A}_{y,\alpha}$ for large y in hand, we are now prepared to implement the large primes upper bound method. Using the single large primes upper bound method, we find for $y = 500$, $z = 10^{14}$, and $y_{max} = 5 \times 10^7$, that

$$\mathbf{d}\mathscr{A} \leq 0.247731321\ldots,$$

which is an improvement of about 2.14×10^{-4} over the Deléglise upper bound of $0.247945250\ldots$. With the two large prime upper bound method, this improves to

$$\mathbf{d}\mathscr{A} \leq 0.247665510\ldots,$$

which is an additional improvement over the single prime calculation of about 6.58×10^{-5}.

4.3 The hybrid algorithm

We will refer to the combination of all of the ideas presented in this chapter, with the exception of the small primes method, as the *hybrid algorithm*. Using this hybrid algorithm, we arrive at the following upper and lower bounds for $\mathbf{d}\,\mathscr{A}$, where $y = 500$, $z = 10^{14}$, and $p_{\max} = 5 \times 10^7$:

$$0.247616464 \le \mathbf{d}\,\mathscr{A} \le 0.247656571.$$

The difference between the upper and lower bounds is about 4.01×10^{-5}.

For comparison, we display the original Deléglise bounds,

$$0.247451383 \le \mathbf{d}\,\mathscr{A} \le 0.247945251,$$

for which the difference between the upper and lower bounds is about 4.93×10^{-4}.

We summarize our results in a table.

Method	Lower	Upper	Difference
Deléglise	0.2474513	0.2479453	4.94×10^{-4}
Small 2	0.2474605	"	4.85×10^{-4}
Small 2, 3	0.2474610	"	4.84×10^{-4}
Medium	0.2474757	"	4.70×10^{-4}
1 large	0.2475747	0.2477314	1.57×10^{-4}
1 lg, med	0.2475991	"	1.32×10^{-4}
2 large	0.2475921	0.2476566	6.45×10^{-5}
2 lg, med	0.2476164	"	4.02×10^{-5}

If we now increase the value of z, we expect improved bounds for the density of abundants. Choosing $z = 10^{15}$ and $y = 500$, we first use Deléglise's original algorithm for comparison. This gives the bounds

$$0.2474678 < \mathbf{d}\mathscr{A} < 0.2479570,$$

where the difference between the upper and lower bounds is about 4.89×10^{-4}. Thus, we see that simply choosing a larger value for z in the Deléglise program does not yield a comparable improvement to the hybrid algorithm at $z = 10^{14}$.

If we use the hybrid algorithm again with $z = 10^{15}$, $y = 500$, and $p_{\max} = 5 \times 10^{7}$, we find the following bounds.

Theorem 4.13. *The density of the set of abundant numbers has the bounds*

$$0.2476171 < \mathbf{d}\mathscr{A} < 0.2476475,$$

with a difference between the upper and lower bounds of 3.04×10^{-5}. *Thus,*

$$\mathbf{d}\,\mathscr{A} = 0.2476\ldots.$$

Finally, we include below the C++ code used to compute the results of the hybrid method. The code is based on Deléglise's, and is in fact a modification of his original code which he has generously provided. In particular, his backtracking algorithm to identify y-smooth numbers up to z has been left unchanged.

```
// abund11.cc
// Version 1.1 - the hybrid algorithm

#include<iostream>
#include<fstream>
#include<iomanip>
#include<cmath>

using namespace std;
typedef long long Long;

const Long PBD = 50000000; // upper bound for calculated primes
const Long KBD = 3001134;  // pi(PBD)

// e^{-\gamma}
const long double eneggam = 0.561459483566885169824143214
8;
```

```
long double F[KBD+1]; // F(k)
// upper and lower bounds for F(k)
long double Fupper(Long);
long double Flower(Long);

// upper and lower bounds for A(1/(u-1), k)
long double Aupper(long double, Long);
long double Alower(long double, Long);

// array for asymptotic method
long double AKarr[KBD+1];



long double sumpinv[KBD+1]; // sum of reciprocal primes up to p_k
// upper and lower bounds for sum of reciprocal primes up to x
long double pinvupper(long double x);
long double pinvlower(long double x);

Long pi[PBD+1]; // pi(x)
// upper and lower bounds for pi(x)
Long piupper(long double x);
Long pilower(long double x);

Long prime[KBD+1]; // p_k
```

```
// upper and lower bounds for p_k
Long primeupper(Long k);
Long primelower(Long k);

long double L[5][KBD+1]; // reduced moments

// default values for K, Y, N
int K=95;
Long Y=500;
Long N=1000;

long double Fy; // F(Y)

// bounds for large primes method
long double b1=2.0*(1.0-1.0/(Y+1.0));
long double b2=b1*(1.0-1.0/(Y+1.0));

long double suminv_nab=1.0; // backtracking doesn't include n=1.

// double primes sum
long double p2uppersum=0.0;
long double p2lowersum=0.0;

// small primes sum
```

```
long double suminv_ab1=0.0; // n<=z/2

long double suminv_ab2=0.0; // n> z/2

long double suminv_ab3=0.0; // n> z/3 and odd


// medium primes sum

long double suminv_med=0.0;


// stacks used by backtracking

Long a[KBD+1];  // a[k] = exponent of prime(k)

Long Pk[KBD+1]; //Pk[k] = prime(k)^a[k]

Long Sk[KBD+1]; // Sk[k] = sum(prime(k), j=0..k)

Long sigma[KBD+1]; // sigma[k] = prod(Sk[j], j=1..k)


// string to Long converter

Long atoll(char *str)

{

    Long zval;


    zval = 0;

    for (; *str; str++)

    {

        zval = 10*zval + (*str - '0');

    }

    return zval;
```

```
}

void traite(int k, Long n)
{
    long double invn=1.0/n;
    Sk[k] += Pk[k];
    sigma[k] = sigma[k-1]*Sk[k];
    long double sigmak=sigma[k];

    if (sigmak >= 2 * n) // This n is nondeficient
    {
        // small primes method
        if(n*2<=N)
            suminv_ab1+=invn;
        else
            suminv_ab2+=invn;
        if(a[1]==0 && n*3>N)
            suminv_ab3+=invn;

        // medium primes method
        if(n*Y<=N)
        {
            suminv_med+=invn*Fy;
        }
```

```
    else
    {
        suminv_med+=invn*Flower((prime[k]*n>N)?k:piupper
            ((Long)ceil(N*invn)));// max of k and pi(z/n).
    }
}
else // This n is deficient
{
    suminv_nab+=invn;

    // double large prime method
    long double a1 = ((long double)sigmak)/(2.0 * n-sigmak);
    long double a2 = a1+sqrt(a1*(1.0+a1));
    long double p2lower=0.0;
    long double p2upper=0.0;

    if(sigmak>b2*n)
    {
        Long i;
        Long piua1=piupper(a1);
        Long pila1=pilower(a1);
        Long piua2=piupper(a2);
        Long pila2=pilower(a2);
        if(K<piua2)
```

```
        {
            p2lower+=Fy-Fupper(pila2);
            p2upper+=Fy-Flower(piua2)+Aupper(a1, pila2);

            for(i=((K+1<pila1)?pila1:K+1);
                i<=((KBD<piua2)?KBD:piua2);i++)
            {
                long double pl=(long double)primelower(i);
                long double pu=(long double)primeupper(i);
                Long hnpl=(Long)floor(sigmak*(pu+1.0)
                    /(2.0*n*pu-sigmak*(pu+1.0)));
                if(hnpl>0)
                {
                    p2lower -= Fupper(pilower(hnpl))/(pl-1.0);
                }
            }

            for(i=((K+1<piua1)?piua1:K+1);
                i<=((KBD<pila2)?KBD:pila2);i++)
            {
                long double pl=(long double)primelower(i);
                long double pu=(long double)primeupper(i);
                Long hnpl=(Long)floor(sigmak*(pu+1.0)
                    /(2.0*n*pu-sigmak*(pu+1.0)));
```

```
        Long hnpu=(Long)floor(sigmak*(pl+1.0)
            /(2.0*n*pl-sigmak*(pl+1.0)));
        if(hnpu>0)
        {
            p2upper-=(Flower(piupper(hnpu))
                -Aupper(hnpu,pilower(hnpl)))/(pu-1.0);
        }
    }

    if(KBD<piua2)
    {
        long double p=(long double) PBD;
        if(KBD<pila1)
            p2lower-=Fupper(pila2)*(pinvupper(a2)
                -pinvlower(a1)+1.0/a1-1.0/a2);
        else
            p2lower-=Fupper(pila2)*(pinvupper(a2)
                -sumpinv[KBD]+1.0/p-1.0/a2);

        p2upper-=(Flower(piua2)-Aupper(1.0/(2.0*n/sigmak*p/
           (p+1.0)-1.0),pilower(sigmak*(1.0+1.0/a2)/(2.0*n
           -sigmak*(1.0+1.0/a2)))))*(pinvlower(a2)-sumpinv
           [KBD])*(1.0+1.0/((longdouble)p-1.0));
    }
```

```
            }
        }
        p2lowersum+=p2lower*invn;
        p2uppersum+=p2upper*invn;

        if(sigmak*(1.0+1.0/prime[K+1])*(1.0+1.0/prime[K+2])<2*n)
        {
            p2uppersum+=Aupper(a1,K)*invn;
        }
    }
}

// Backtracking computes all p_K-smooth integers up to n (except 1)
void back(int k, Long n) {
    Long nextn;
    nextn = n;
    while (nextn <= N)
    {
        if(a[k])
        {
            traite(k,nextn); // For computing bounds for A(2)
        }
        if ((k < K) and (nextn*prime[k+1] <= N))
        //Take care of overflow
```

```
        {
            a[k+1]=0;
            Pk[k+1] = 1;
            Sk[k+1] = 1;
            sigma[k+1] = sigma[k];
            back(k+1,nextn);
        }
        a[k]++;
        nextn = nextn * prime[k];
        Pk[k] *= prime[k];
    }
}

void initprimes()
{
    // identify primes up to PBD:
    long int rootn;
    rootn=(long int)floor(sqrt(PBD));

    int interval[rootn+1];
    long int i;

    interval[0]=0;
    interval[1]=0;
```

```
for( i=2; i<=rootn; i++)
{
    interval[i]=1;
}


long int ptr=2;


while(ptr<=floor(sqrt(rootn)))
{
    for(i=ptr*ptr; i<=rootn; i=i+ptr)
    {
        interval[i]=0;
    }


    ptr++;
    while(interval[ptr]==0) ptr++;
}


long int pcount=0;
long int count;


for(i=0; i<=rootn; i++)
{
    pcount+=interval[i];
```

```
}

// fill prime[], pi[], and F[] arrays:
prime[0]=1;
pi[0]=0;
pi[1]=0;
F[0]=1.0;

i=1;
long int j;
for(j=2; j<=rootn; j++)
{
    pi[j]=pi[j-1]+interval[j];
    if(interval[j]==1)
    {
        prime[i]=j;
        F[i]=F[i-1]*(prime[i]-1.0)/prime[i];
        i++;
    }
}

// sieve intervals of length rootn

long int m, r;
```

```
m=PBD/rootn;
r=PBD-m*rootn;

count = pcount;  // count the first interval

for(i=1; i<m;i++)
{
    interval[0]=0;
    for(j=1; j<=rootn; j++)
    {
        interval[j]=1;
    }

    for(j=1;j<=pcount;j++)
    {
        for(ptr=prime[j]-((i*rootn)%prime[j]);ptr<=rootn;
            ptr+=prime[j])
        {
            interval[ptr]=0;
        }
    }

    for(j=0; j<=rootn; j++)
```

```
        {
            pi[rootn*i+j]=pi[rootn*i+j-1]+interval[j];
            if(interval[j]==1)
            {
                count++;
                prime[count]=rootn*i+j;
                F[count]=F[count-1]*(prime[count]-1.0)/prime[count];
            }
        }
    }

    // last interval

    interval[0]=0;
    for(j=1; j<=r; j++)
    {
        interval[j]=1;
    }
    for(j=r+1; j<=rootn; j++)
    {
        interval[j]=0;
    }

    for(j=1;j<=pcount;j++)
```

```
{
    for(ptr=prime[j]-((m*rootn)%prime[j]);ptr<=r;ptr+=prime[j])
    {
        interval[ptr]=0;
    }
}



for(j=0; j<=r; j++)
{
    pi[rootn*i+j]=pi[rootn*i+j-1]+interval[j];
    if(interval[j]==1)
    {
        count++;
        prime[count]=rootn*i+j;
        F[count]=F[count-1]*(prime[count]-1.0)/prime[count];
    }
}

// calculate sum of reciprocal primes

sumpinv[0]=0.0;
for(j=1;j<=KBD;j++)
{
```

```
        sumpinv[j]=sumpinv[j-1]+1.0/prime[j];
    }

    // calculate reduced moments

    L[1][KBD]=1.0706484444688249785754788801621464049769e-9;
    L[2][KBD]=1.2125360381904000849879330046048765336008e-17;
    L[3][KBD]=1.8416871110871013722627356991238754238205e-25;
    L[4][KBD]=3.29594356911638569265174956894439063553e-33;

    for(j=KBD-1;j>=0;j--)
    {
        long double p1=1.0*prime[j+1];
        long double p2=p1*p1;
        long double p3=p2*p1;
        long double p4=p3*p1;
        long double p5=p4*p1;
        long double p10=p5*p5;
        long double l1=1.0/(p2-1.0);
        long double l2=(1.0+p2)/(p2-1.0)/(p3-1.0);
        long double l22=(2.0*p3+p2-1.0)/(p3-1.0)/(p2-1.0);
        long double l21=p2/(p3-1.0)/(p1-1.0);
        long double l3=(p4-p3+3.0*p2-p1+1.0)/(p1-1.0)/(p3-1.0)
            /(p4-1.0);
```

```
        long double l33=(3.0*p3*p3+p4-p3-p1+1.0)/(p1-1.0)/(p3-1.0)
            /(p4-1.0);
        long double l32=p3*(p3+p2+1.0)/(p1-1.0)/(p3-1.0)/(p4-1.0);
        long double l31=p2*(p3+p1+1.0)/(p1-1.0)/(p3-1.0)/(p4-1.0);
        long double l4=(p4*p4-p3*p4+4.0*p2*p4+p5+2.0*p4+p3+4.0*p2-p1+1)
            /(p1-1.0)/(p3-1.0)/(p4-1.0)/(p5-1.0);
        long double l44=(4.0*p10*p1+2.0*p10+2.0*p5*p4+p4*p4+3.0*p4*p3
            -2.0*p3*p3+p5+p3+p1-1.0)/(p1-1.0)/(p3-1.0)/(p4-1.0)
            /(p5-1.0);
        long double l43=p4*(p4*p3+2.0*p3*p3+p5+2.0*p4+3.0*p3+2.0*p2+1.0)
            /(p1-1.0)/(p3-1.0)/(p4-1.0)/(p5-1.0);
        long double l42=p3*(p3*p3+p4+2.0*p3+p2+1.0)/(p1-1.0)/(p3-1.0)
            /(p3-p2+p1-1.0)/(p5-1.0);
        long double l41=p2*(p4*p3+2.0*p5+3.0*p4+2.0*p3+p2+2.0*p1+1.0)
            /(p1-1.0)/(p3-1.0)/(p4-1.0)/(p5-1.0);


        L[1][j]=L[1][j+1]+l1+L[1][j+1]*l1;
        L[2][j]=L[2][j+1]+l2+L[2][j+1]*l22+2.0*L[1][j+1]*l21;
        L[3][j]=L[3][j+1]+l3+L[3][j+1]*l33+3.0*L[2][j+1]*l32
            +3.0*L[1][j+1]*l31;
        L[4][j]=L[4][j+1]+l4+L[4][j+1]*l44+4.0*L[3][j+1]*l43
            +6.0*L[2][j+1]*l42+4.0*L[1][j+1]*l41;
    }
```

```
// initialize AKarr, the asymptotic bound


long double c=3.222;
for(i=1;i<=KBD;i++)
{
    long double tmax=floor(log((long double)PBD/prime[i])/log(c));
    long double fact=1.0;
    long double sum=0.0;
    long double logct=log(pow(c,tmax+1.0)*prime[i]);
    long double logctm1=log(pow(c,tmax)*prime[i]);
    long double ubd=log(1.0+log(c)/logctm1) + 1.0/10.0/logct/logct
        + 4.0/15.0/logct/logct/logct + 1.0/10.0/logctm1/logctm1
        + 4.0/15.0/logctm1/logctm1/logctm1;


    for(j=1;j<=tmax;j++)
    {
        fact*=j;
        sum+=(pow((long double)sumpinv[pi[(Long)floor(pow(c,
            (long double)j)*prime[i])]]-sumpinv[pi[(Long)pow(c,
            (long double)j-1)*prime[i]]],(long double)j)-pow(
            (long double)ubd,(long double)j))/fact;
    }
    sum+=exp(ubd)-1.0;
    AKarr[i]=F[i]*sum;
```

```
    }
}

Long pilower(long double x) {
    if(x<=PBD) return pi[(Long)floor(x)];
    else if(x<= 50096009)
        return pi[PBD];
    else
    {
        long double lx=log(x);
        return (Long) floor(x/lx*(1.0+1.0/lx+1.8/lx/lx));
    }
}

Long piupper(long double x) {
    if(x<=PBD) return pi[(Long)floor(x)];
    else
    {
        long double lx=log(x);
        return (Long) ceil(x/lx*(1.0+1.0/lx+2.51/lx/lx));
    }
}

Long primelower(Long k) {
```

```
    if(k<=KBD) return prime[k];
    else
    {
        long double lk=log(k);
        return (Long)floor(k*(lk+log(lk)-1.0+(log(lk)-2.25)/lk));
    }
}

Long primeupper(Long k) {
    if(k<=KBD) return prime[k];
    else
    {
        long double lk=log(k);
        return (Long)ceil(k*(lk+log(lk)-1.0+(log(lk)-1.8)/lk));
    }
}

long double pinvupper(long double x) {
    if(x<=PBD)
    {
        return sumpinv[pi[(Long)floor(x)]];
    }
    else // Dusart upper bound valid for x>=10372
    {
```

```
        long double lx=log(x);
        return log(lx) + 0.261497212847643 + (1.0/10.0 + 4.0/15.0/lx)
            /lx/lx;
    }
}

long double pinvlower(long double x) {
    if(x<=PBD)
    {
        return sumpinv[pi[(Long)floor(x)]];
    }
    else // Dusart lower bound valid for x>1
    {
        long double lx=log(x);
        return log(lx) + 0.261497212847643 - (1.0/10.0 + 4.0/15.0/lx)
            /lx/lx;
    }
}

long double Flower(Long k)// k=index of prime p_k. {
    if(k<=KBD) // use array F
        return F[k];
    else // k> KBD, so use Dusart
    {
```

```
        long double lpk=log(primeupper(k));
        return eneggam*(1.0-0.2/lpk/lpk)/lpk;
    }
}

long double Fupper(Long k)// k=index of prime p_k. {
    if(k<=KBD) // use array F
        return F[k];
    else if(k<=3035782) // F[KBD] is smaller than Dusart for k<=3033524,
    // but we use a lower bound for pk, so k<=3035782.
    {
        return F[KBD];
    }
    else // k> 3035782, so Dusart is smaller
    {
        long double lpk=log(primelower(k));
        return eneggam*(1.0+0.2/lpk/lpk)/lpk;
    }
}

long double Alower(long double u_1inv, Long k) // u_1inv=1/(u-1),
k=index of prime p_k. {
    if(u_1inv<=0) // Aupper=Fupper
    {
```

```
        return Fupper(k);
    }
    long double bound=Flower(k)-Fupper(pilower(u_1inv));
    if(bound<=0)
    {
        return 0;
    }
    return bound;
}

long double Aupper(long double u_1inv, Long k)// u_1inv=1/(u-1),
k=index of prime p_k. {
    if(k<=KBD)
    {
        long double min = 1.0;
        long double u_0inv=u_1inv;

        long double y = L[1][k]*u_0inv;
        if (y < min)
          min = y;
        u_0inv*=u_1inv;
        y = L[2][k]*u_0inv;
        if (y < min)
            min = y;
```

```
        u_0inv*=u_1inv;
        y = L[3][k]*u_0inv;
        if (y < min)
              min = y;
        u_0inv*=u_1inv;
        y = L[4][k]*u_0inv;
        if (y < min)
          min = y;
        Long piflu=pilower(u_1inv);
        if(piflu<=KBD && k<=piflu)
        {
            y=F[k]-F[piflu]+AKarr[piflu];
            if(y<min*F[k])
                return y;
        }
        return F[k]*min;
    }
    else // use an upper bound for the first moment
    {
        if(k<3388888) // if pi(k)<56855672, then L[1][KBD] is better
            return Fupper(k)*L[1][KBD]*u_1inv;
        else
        {
            long double y=k*(log(k)+log(log(k))-1+(log(log(k))-2.25)
```

```
                /log(k));
            long double f1=1./log(y)+1.5524/(log(y)*log(y));
            return Fupper(k)*(f1/y*(+1+f1/y)+0.0067/(y*y*y))*u_1inv;
        }
    }
}


void init(int argc, char* argv[]) {
    if(argc > 1) N  = atoll(argv[1]);
    if(argc > 2) K = atoll(argv[2]);
    cout << "K = " << K << endl;
    cout << "N = " << N << endl;
    cout << "KBD = " << KBD << endl;
    sigma[0] = 1;
    Pk[1] = 1;
    Sk[1] = 1;
    sigma[1] = 1;
    cout << setprecision(20);

    initprimes();
    Y = prime[K];
    Fy = F[K];

    // bounds for large prime method
```

```
    b1=2.0*(1.0-1.0/(Y+1.0));
    b2=b1*(1.0-1.0/(Y+1.0));
}

int main(int argc, char* argv[]) {
    init(argc,argv);
    int i;

    back(1,1);

    cout << "p2uppersum = " << p2uppersum << endl;
    cout << "p2lowersum = " << p2lowersum << endl;
    cout << "suminv_ab1 = " << suminv_ab1 << endl;
    cout << "suminv_ab2 = " << suminv_ab2 << endl;
    cout << "suminv_ab3 = " << suminv_ab3 << endl;
    cout << "suminv_med = " << suminv_med << endl;

    cout << "2p + small lower = " << p2lowersum+(suminv_ab1 +
        2.0*suminv_ab2+ 0.5*suminv_ab3)*F[K] << endl;
    cout << "2p + medium lower = " << p2lowersum+suminv_med << endl;
    cout << "2p upper = " << 1.0-F[K]*suminv_nab + p2uppersum << endl;
}
```

Chapter 5

The α-pnd method

When we introduced the ideas of Behrend in Section 2.3, we noted that Behrend's lower bound idea used a certain set of nondeficient numbers. We will introduce this set, called the primitive nondeficient numbers, along with their α-generalizations, the α-primitive nondeficient numbers, in the next section. Just as Deléglise was able to create both an upper and lower bound algorithm from Behrend's upper bound idea, we will be able to extend Behrend's lower bound method into a method to calculate both upper and lower bounds for the density of α-abundant numbers. We call this the α-pnd method.

5.1 Primitive nondeficient numbers

We will introduce the idea of primitive non-deficient numbers by considering the sequence of non-deficient numbers:

$$6, 12, 18, 20, 24, 28, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100, \ldots.$$

It may be noticed that many of these are multiples of 6, the first perfect number. In fact, every multiple of 6 up to 96 appears. Does this trend continue? If so, then a number of other questions naturally arise. First, are all multiples of each perfect number on this list? Second, are there numbers on this list that are not multiples of perfect numbers? We can answer this one right away. We have an example in the fourth entry in the list above: Note that 20 is not perfect, and no divisor of 20 is perfect. But the first few multiples of 20 are also on the list, again leading us to ask the same question for this number: Are all multiples of 20 on this list?

The answers to these questions leads naturally to the consideration of a particular subset of non-deficient numbers, which we will call primitive. This subset will be useful in giving us an alternative method of estimating the density of abundant numbers. We will now show that the line of inquiry concerning multiples of non-deficient numbers is a productive one. The following lemma was essentially proven using Lemma 2.1 in the discussion following that lemma, but we will prove it directly below.

Lemma 5.1. *Let n and m be natural numbers. In particular*

$$h(nm) \geq h(n),$$

with equality only when $m = 1$.

Proof. The case when $m = 1$ is clear. When $m \neq 1$, we use that

$$h(n) = \sum_{d|n} \frac{1}{d},$$

so that

$$h(nm) = \sum_{d|nm} \frac{1}{d} \geq \sum_{d|n} \frac{1}{d} + \frac{1}{nm} > h(n).$$

This proves the lemma. □

Thus, we see that not only are proper multiples of each perfect number abundant, it is also the case in general that proper multiples of any non-deficient number is abundant.

Now let us return to the list of non-deficients. Since we now know that all of the multiples of 6 occur, let us remove these and see what is left:

$$20, 28, 40, 56, 70, 80, 88, 100, \ldots .$$

Of the remaining numbers, many are multiples of the first number, 20. Removing these, we are left with

$$28, 56, 70, 88, \ldots .$$

If we continue this process, we find that all non-deficient numbers up to 100 are multiples of 6, 20, 28, 70, or 88. It may be realized by now that we are using a procedure analogous to the Sieve of Eratosthenes, but on a different set. In general, we will use the term primitive for such a set of numbers which correspond to the primes in this way.

Definition 5.2. If $\mathscr{S}$ is a subset of the natural numbers, let $\mathscr{M}(\mathscr{S})$ denote the set

$$\mathscr{M}(\mathscr{S}) := \{sn : s \in \mathscr{S}, n \in \mathbb{N}\}.$$

We call $\mathscr{M}(\mathscr{S})$ the *set of multiples* of $\mathscr{S}$ and say that a set $\mathscr{S}$ *generates* set $\mathscr{T}$ if

$\mathscr{T} = \mathscr{M}(\mathscr{S})$.

There is a unique minimal generating set for $\mathscr{T}$ which can be found by taking the intersection of all $\mathscr{S}$ such that $\mathscr{T} = \mathscr{M}(\mathscr{S})$. To see this, it suffices to show that if $\mathscr{T} = \mathscr{M}(\mathscr{S}_i)$ for $i = 1, 2$, then $\mathscr{T} = \mathscr{M}(\mathscr{S}_1 \cap \mathscr{S}_2)$. One inclusion is easy, since $\mathscr{M}(\mathscr{S}_1 \cap \mathscr{S}_2) \subseteq \mathscr{M}(\mathscr{S}_1) = \mathscr{T}$. To see the opposite inclusion, suppose $t \in \mathscr{T}$. Let s_i be the smallest member $s \in \mathscr{S}_i$ such that $ms = t$, and let $m_i s_i = t$. Since $s_1 \in \mathscr{S}_1 \subseteq \mathscr{M}(\mathscr{S}_2)$, there are m and $s \in \mathscr{S}_2$ such that $ms = s_1$. Thus $s_1 \geq s$. However, by definition of s_2, since $t = m_1 s_1 = (m_1 m)s$, we have that $s \geq s_2$, so we conclude that $s_1 \geq s_2$. However, this argument is symmetric in switching subscripts 1 and 2, so $s_2 \geq s_1$. We conclude that $s_1 = s_2 \in \mathscr{S}_1 \cap \mathscr{S}_2$. Thus $t \in \mathscr{M}(\mathscr{S}_1 \cap \mathscr{S}_2)$.

By analogy with the primes, we will use the adjective *primitive* when referring to either the unique minimal generating set or its members. In view of Lemma 5.1, we have that a *primitive non-deficient number* (pnd) is a number n such that $h(n) \geq 2$, while for each proper divisor d of n, $h(d) < 2$. Analogously, for any real number α a *primitive α-non-deficient number* (α-pnd) is a number n such that $h(n) \geq \alpha$ while for each proper divisor d, $h(d) < \alpha$. The set of pnd's and α-pnd's will be denoted $\mathbb{P}$ and $\mathbb{P}_\alpha$, respectively. Thus, we have the following proposition.

Proposition 5.3. *The sets $\mathscr{A}'$ and $\mathscr{A}'_\alpha$ are generated by $\mathbb{P}$ and $\mathbb{P}_\alpha$, respectively, so that*

$$\mathscr{A}' = \mathscr{M}(\mathbb{P}) \qquad \text{and} \qquad \mathscr{A}'_\alpha = \mathscr{M}(\mathbb{P}_\alpha).$$

Remark 5.4. In the literature primitive non-deficient numbers are often called primitive abundant numbers, and are abbreviated pan. This definition becomes the natural one provided an abundant number n is redefined so that $h(n) \geq 2$, as is done in [3, 5]. However, we will not be following these conventions.

5.2 A density lower bound method

In Behrend's doctoral dissertation, a lower bound for the lower density of abundant numbers is found by calculating the density of the multiple set of a finite subset of the primitive non-deficient numbers. (We say lower density since the density of abundant numbers was not yet known to exist.) For instance, at one point Behrend requires the density of the multiple set of four pnd's:

$$\mathbf{d}\,\mathscr{M}(\{2 \cdot 3,\ 2^2 \cdot 5,\ 2^2 \cdot 7,\ 2 \cdot 5 \cdot 7\}).$$

To find this density, he partitions the multiple set into disjoint subsets in the following way. Define A_a^c to be the set of numbers that are multiples an of a such that $(n, c) = 1$. Then he observes that the density of such a set has the simple form

$$\mathbf{d}\,A_a^c = \frac{\phi(c)}{c} \cdot \frac{1}{a}.$$

By choosing appropriate values of c corresponding to each pnd a such that the sets A_a^c are disjoint, their individual densities can be evaluated. Then summing these densities gives us the total density. We seek to generalize this method so that, given an arbitrary set of primitive non-deficient numbers, we can identify for each pnd a an appropriate c such that $\sum_a \mathbf{d}\,A_a^c$ gives the density of the multiples of these pnd's.

Suppose we begin naïvely by considering consecutive pnd's. For the pair of pnd's 6 and 20, we observe that their multiples coincide at the multiples of $[6, 20]$, so we write the combined density of multiples of 6 and 20 as

$$\frac{1}{6} + \frac{1}{20} - \frac{1}{[6,20]} = \frac{1}{6} + \frac{1}{20}\left(1 - \frac{20}{[6,20]}\right) = \frac{1}{6} + \frac{1}{20}\left(1 - \frac{(6,20)}{6}\right).$$

Indeed, for any two numbers a_1 and a_2 we have

$$\begin{aligned} d(a_1\mathbb{N} \cup a_2\mathbb{N}) &= da_1\mathbb{N} + da_2\mathbb{N} - d[a_1, a_2]\mathbb{N} \\ &= \frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{[a_1, a_2]} = \frac{1}{a_1} + \frac{1}{a_2} - \frac{(a_1, a_2)}{a_1 a_2} \\ &= \frac{1}{a_1} + \frac{1}{a_2}\left(1 - \frac{(a_1, a_2)}{a_1}\right) = \frac{1}{a_1} + \frac{1}{a_2}\left(1 - \frac{1}{c_{1,2}}\right), \end{aligned}$$

where $c_{1,2} = a_1/(a_1, a_2)$. We can think of $1 - \frac{1}{c_{1,2}}$ as a correction factor that takes into account the overlap that the multiples of $a_2\mathbb{N}$ have with $a_1\mathbb{N}$.

Unfortunately, for three terms the general correction factor is not as clean. In fact, the additional term which is the density of the set $a_3\mathbb{N} \setminus (a_1\mathbb{N} \cup a_2\mathbb{N})$ can be written

$$\begin{aligned} \mathbf{d}\left(a_3\mathbb{N} \setminus (a_1\mathbb{N} \cup a_2\mathbb{N})\right) &= \frac{1}{a_3} - \frac{1}{[a_1, a_3]} - \frac{1}{[a_2, a_3]} + \frac{1}{[a_1, a_2, a_3]} \\ &= \frac{1}{a_3} - \frac{(a_1, a_3)}{a_1 a_3} - \frac{(a_2, a_3)}{a_2 a_3} + \frac{([a_1, a_2], a_3)}{[a_1, a_2] a_3} \\ &= \frac{1}{a_3}\left(1 - \frac{(a_1, a_3)}{a_1} - \frac{(a_2, a_3)}{a_2} + \frac{[(a_1, a_3), (a_2, a_3)](a_1, a_2)}{a_1 a_2}\right) \\ &= \frac{1}{a_3}\left(1 - \frac{(a_1, a_3)}{a_1} - \frac{(a_2, a_3)}{a_2} + \frac{(a_1, a_3)(a_2, a_3)(a_1, a_2)}{a_1 a_2((a_1, a_3), (a_2, a_3))}\right). \end{aligned}$$

Now suppose that we have

$$(a_1, a_2) = ((a_1, a_3), (a_2, a_3)). \tag{5.1}$$

Then we arrive at a simpler expression for the correction factor, since

$$1 - \frac{(a_1, a_3)}{a_1} - \frac{(a_2, a_3)}{a_2} + \frac{(a_1, a_3)(a_2, a_3)}{a_1 a_2} = \left(1 - \frac{(a_1, a_3)}{a_1}\right)\left(1 - \frac{(a_2, a_3)}{a_2}\right).$$

It is easy to see that the condition (5.1) required for this simplification is equivalent to the condition

$$\left(\frac{a_1}{(a_1, a_3)}, \frac{a_2}{(a_2, a_3)}\right) = 1. \tag{5.2}$$

The following proposition generalizes the observations made above to any number of terms. For convenience, we use the following notation.

Definition 5.5. For sequence $(a_i)_{i=1}^k$ we write $\mathscr{M}_j(a_1, a_2, \ldots, a_k)$ for the set of multiples of a_j that are not multiples of any a_i, $i < j$. Thus,

$$\mathscr{M}_j(a_1, a_2, \ldots, a_k) := a_j\mathbb{N} \setminus \bigcup_{i<j} a_i\mathbb{N}.$$

We note that this allows us to partition $\mathscr{M}(\{a_1, a_2, \ldots, a_k\})$ into the disjoint union of subsets $\mathscr{M}_j(a_1, a_2, \ldots, a_k)$, $j \le k$.

Proposition 5.6. *Let $A = (a_j)_{j=1}^k$ be a sequence of natural numbers and for each j construct the sequence $C_j = (c_{i,j})_{i=1}^{j-1}$ whose elements are defined by $c_{i,j} = a_i/(a_i, a_j)$. In addition suppose that for each j the elements of C_j are pairwise coprime. Then the density of $\mathscr{M}_j(A)$ is given by*

$$\mathbf{d}\,\mathscr{M}_j(A) = \frac{1}{a_j}\prod_{i=1}^{j-1}\left(1 - \frac{1}{c_{i,j}}\right),$$

so that

$$\mathbf{d}\,\mathscr{M}(A) = \sum_{j=1}^{k}\frac{1}{a_j}\prod_{i=1}^{j-1}\left(1 - \frac{1}{c_{i,j}}\right). \tag{5.3}$$

Proof. We first write

$$\mathbf{d}\left(\bigcup_{i=1}^{j} a_i\mathbb{N}\right) = \mathbf{d}\left(\bigcup_{i=1}^{j-1} a_i\mathbb{N} \sqcup \left(a_j\mathbb{N} \setminus \bigcup_{i=1}^{j-1} a_i\mathbb{N}\right)\right)$$
$$= \mathbf{d}\left(\bigcup_{i=1}^{j-1} a_i\mathbb{N}\right) + \mathbf{d}\left(a_j\mathbb{N} \setminus \bigcup_{i=1}^{j-1} a_i\mathbb{N}\right).$$

Thus, to prove the proposition, it suffices to show that

$$\mathbf{d}\left(\bigcup_{i=1}^{j} a_i\mathbb{N}\right) - \mathbf{d}\left(\bigcup_{i=1}^{j-1} a_i\mathbb{N}\right) = \mathbf{d}\left(a_j\mathbb{N} \setminus \bigcup_{i=1}^{j-1} a_i\mathbb{N}\right) = \frac{1}{a_j}\prod_{i=1}^{j-1}\left(1 - \frac{1}{c_{i,j}}\right). \qquad (*)$$

We prove this by induction on j. When $j = 1$, $(*)$ becomes the equation $\mathbf{d}\, a_1\mathbb{N} = 1/a_1$, which is true. Suppose the relation $(*)$ is true for $j = k-1$. Then by the induction hypothesis the density of the set S of multiples of a_k not divisible by a_i for $i \leq k-2$ is

$$\mathbf{d}\, S = \mathbf{d}\left(a_k\mathbb{N} \setminus \bigcup_{i=1}^{k-2} a_i\mathbb{N}\right) = \frac{1}{a_k}\prod_{i=1}^{k-2}\left(1 - \frac{1}{c_{i,k}}\right).$$

We want to subtract the density of multiples of a_{k-1} in S from the density of S. Thus, we want the density of the set

$$T := a_{k-1}\mathbb{N} \cap S = a_{k-1}\mathbb{N} \cap a_k\mathbb{N} \cap \left(\bigcup_{i=1}^{k-2} a_i\mathbb{N}\right)^c = [a_{k-1}, a_k]\mathbb{N} \setminus \bigcup_{i=1}^{k-2} a_i\mathbb{N}.$$

Now we write $a'_{k-1} := [a_k, a_{k-1}]$ and $c'_{i,k-1} := a_i/(a_i, a'_{k-1})$. Since $a_k \mid a'_{k-1}$, we have $c'_{i,k-1} \mid c_{i,k}$. This in turn gives that $(c_{i_1,k}, c_{i_2,k}) = 1$ implies $(c'_{i_1,k-1}, c'_{i_2,k-1}) = 1$. This allows us to use the induction hypothesis on T so that we have

$$\mathbf{d}\, T = \frac{1}{[a_k, a_{k-1}]}\prod_{i=1}^{k-2}\left(1 - \frac{1}{c'_{i,k-1}}\right) = \frac{1}{a_k c_{k-1,k}}\prod_{i=1}^{k-2}\left(1 - \frac{1}{c'_{i,k-1}}\right).$$

We will show that $c_{i,k} = c'_{i,k-1}$ for each $i \le k-2$. Then

$$\mathbf{d}\,T = \frac{1}{a_k c_{k-1,k}} \prod_{i=1}^{k-2} \left(1 - \frac{1}{c_{i,k}}\right),$$

so that the difference $\mathbf{d}\,S - \mathbf{d}\,T$ becomes

$$\frac{1}{a_k} \prod_{i=1}^{k-2} \left(1 - \frac{1}{c_{i,k}}\right) - \frac{1}{a_k c_{k-1,k}} \prod_{i=1}^{k-2} \left(1 - \frac{1}{c_{i,k}}\right) = \frac{1}{a_k} \prod_{i=1}^{k-1} \left(1 - \frac{1}{c_{i,k}}\right),$$

which proves our result.

Thus, it remains to show that $c_{i,k} = c'_{i,k-1}$ for each $i \le k-2$. We will now use our condition that for each $i \le k-2$,

$$(c_{i,k}, c_{k-1,k}) = \left(\frac{a_i}{(a_i, a_k)}, \frac{a_{k-1}}{(a_{k-1}, a_k)}\right) = 1.$$

For p prime, define e_i for each $i \le k$ such that $p^{e_i} \| a_i$. Then the coprimality condition translates to

$$\min\{e_i - \min\{e_i, e_k\}, e_{k-1} - \min\{e_{k-1}, e_k\}\} = 0.$$

We now consider each of the cases where either the first or second entry is smaller. If the first entry is smaller than the second, then the first entry must be 0 so $\min\{e_i, e_k\} = e_i$ and $e_i \le e_k$. If the second entry is smaller than the first, then we have analogously that $e_{k-1} \le e_k$. Observe that in either of these cases, we have

$$\min\{e_i, \max\{e_k, e_{k-1}\}\} = \min\{e_i, e_k\},$$

which translates to

$$(a_i, [a_k, a_{k-1}]) = (a_i, a_k).$$

Then

$$c'_{i,k-1} = \frac{a_i}{(a_i, [a_k, a_{k-1}])} = \frac{a_i}{(a_i, a_k)} = c_{i,k},$$

as claimed. □

Remark 5.7. In the event that the sequence contains repeated terms, say $a_i = a_j$ for $i < j$, then the result nevertheless applies since in this case

$$c_{i,j} = \frac{a_i}{(a_i, a_j)} = \frac{a_i}{(a_i, a_i)} = 1,$$

so $1 - 1/c_{i,j} = 0$ and $\mathbf{d}\,\mathscr{M}_j(a_1, \ldots, a_j) = 0$ so the jth term does not contribute to the density sum. In any case, in what follows we will only be concerned with sequences having distinct terms.

Definition 5.8. We call $c_{i,j}$ in Proposition 5.6 the *cofactor* of a_j for a_i, and denote the sequence of cofactors of a_j by C_j.

For our purposes it will be useful to have a weaker version of the coprimality condition on C_j. To take the place of C_j, we define

$$C'_j := \{c \in C_j : c' \in C_j, c' \neq c \implies c' \nmid c\}.$$

Note that whereas C_j is a sequence, C'_j is defined as a set since we will not be concerned with the order of its members nor with any multiplicity which may occur in the terms of the sequence C_j. The set C'_j will be called the *reduced cofactor set for* a_j. We can now state the following theorem.

Theorem 5.9. *Let* $(a_j)_{j=1}^k$ *be a sequence of natural numbers and let* $(C'_j)_{j=1}^k$ *be the corresponding sequence of reduced cofactor sets. Suppose in addition that for each* j *the elements of* C'_j *are pairwise coprime. Then*

$$\mathbf{d}\,\mathscr{M}_j(a_1,\ldots,a_j) = \frac{1}{a_j}\prod_{c\in C'_j}\left(1-\frac{1}{c}\right) \tag{5.4}$$

and

$$\mathbf{d}\,\mathscr{M}(\{a_j\}_{j=1}^k) = \sum_{j=1}^k \frac{1}{a_j}\prod_{c\in C'_j}\left(1-\frac{1}{c}\right). \tag{5.5}$$

Proof. We repeat the argument proving Theorem 5.6 where in place of $(*)$ we now show that

$$\mathbf{d}\left(\bigcup_{j=1}^k a_j\mathbb{N}\right) - \mathbf{d}\left(\bigcup_{j=1}^{k-1} a_j\mathbb{N}\right) = \mathbf{d}\left(a_k\mathbb{N}\setminus\bigcup_{j=1}^{k-1} a_j\mathbb{N}\right) = \frac{1}{a_k}\prod_{c\in C'_k}\left(1-\frac{1}{c}\right). \tag{$*'$}$$

We again use induction on k. As before, when $k=1$ we are done, so assume $(*')$ is true for $k-1$ elements. Then calling S' the set of multiples of a_k not divisible by a_j for $j \le k-2$, we have

$$\mathbf{d}\,S' = \mathbf{d}\left(a_k\mathbb{N}\setminus\bigcup_{j=1}^{k-2} a_j\mathbb{N}\right) = \frac{1}{a_k}\prod_{c\in C''_k}\left(1-\frac{1}{c}\right),$$

where C''_k is the reduced cofactor set for a_k with the sequence $(a_1,\ldots,a_{k-2})$. The set T' of multiples of a_{k-1} in S' is

$$T' := [a_{k-1},a_k]\mathbb{N}\setminus\bigcup_{i=1}^{k-2} a_i\mathbb{N},$$

having density

$$\mathbf{d}\,T' = \frac{1}{[a_k, a_{k-1}]} \prod_{c \in C_k'''} \left(1 - \frac{1}{c}\right) = \frac{1}{a_k c_{k-1,k}} \prod_{c \in C_k'''} \left(1 - \frac{1}{c}\right),$$

where C_k'''' is the reduced cofactor set for $[a_{k-1}, a_k]$ with the sequence $(a_1, \ldots, a_{k-2})$.

We need to show that $dT' = 0$ if and only if there is some i such that $c_{i,k} \mid c_{k-1,k}$. Thus, we write

$$\begin{aligned} c_{i,k} \mid c_{k-1,k} &\iff [a_i, a_k] = c_{i,k} a_k \mid c_{k-1,k} a_k = [a_{k-1}, a_k] \\ &\iff a_i \mid [a_{k-1}, a_k] \\ &\iff (a_i, [a_{k-1}, a_k]) = a_i \\ &\iff \frac{a_i}{(a_i, [a_{k-1}, a_k])} = 1, \end{aligned}$$

so 1 appears as an element of C_k'''' and $dT = 0$. On the other hand, if 1 does not appear in C_k'''', then $c_{i,k} \nmid c_{k-1,k}$ for any $i < k - 1$, so $c_{k-1,k} \in C_k'$.

Finally, we must show that in the latter case $C_k'' = C_k'''$. But we have already shown in the previous proof that the cofactor sets are the same before reduction. Thus, they must be the same after reduction as well. We conclude that the difference $dS' - dT'$ is

$$\frac{1}{a_k} \prod_{c \in C_k''} \left(1 - \frac{1}{c}\right) - \frac{1}{a_k c_{k-1,k}} \prod_{c \in C_k'''} \left(1 - \frac{1}{c}\right) = \frac{1}{a_k} \prod_{c \in C_k'} \left(1 - \frac{1}{c}\right),$$

as asserted. □

5.3 The "significance" of prime powers

We seek to identify a sequence of pnd's that satisfies the conditions of Theorem 5.9. That an arbitrary sequence will not work can be seen in the case of the following sequence of pnd's:

$$2^2 \cdot 5, \quad 2^2 \cdot 7, \quad 2 \cdot 3$$

This gives the numbers 10 and 14 as cofactors for 6. However, 10 and 14 are not relatively prime and neither is divisible by the other.

We will determine an ordering that depends on how much a prime factor of a number n contributes to its abundance, in the following sense. Let $h(n) = \sigma(n)/n$. Writing $n = \prod p_i^{e_i}$, where the primes p_i are distinct and each $e_i \geq 1$, we have by the multiplicativity of h that $h(n) = \prod h(p_i^{e_i})$. Now comparing $h(n)$ to $h(n/p_i)$ we see that they differ by the factor

$$\begin{aligned} \frac{h(n)}{h(n/p_i)} &= \frac{h(n/p_i^{e_i})h(p_i^{e_i})}{h(n/p_i^{e_i})h(p_i^{e_i-1})} = \frac{p_i^{e_i+1}-1}{p_i^{e_i}(p_i-1)} \cdot \frac{p_i^{e_i-1}(p_i-1)}{p_i^{e_i}-1} \\ &= \frac{p_i^{e_i+1}-1}{p_i^{e_i+1}-p_i} = 1 + \frac{p_i-1}{p_i(p_i^{e_i}-1)} = 1 + \frac{1}{p_i\sigma(p_i^{e_i-1})} \\ &= 1 + \frac{1}{\sigma(p_i^{e_i})-1}. \end{aligned}$$

We can now see that the effect that removing a prime factor p from n has on $h(n)$ depends on $\sigma(p^e), p^e || n$, with larger values $\sigma(p^e)$ having a smaller effect. We will now define an ordering on prime powers that reflects this effect. However, it may be the case that more than one prime power have the same sigma value. For instance, 2^4 and 5^2 both have a σ-value of 31. In such a case we will want to distinguish the two prime powers. We thus make the following definition:

Definition 5.10. Suppose there are k prime powers $p_i^{e_i}$ with equal σ-values and with $p_1 < \cdots < p_k$. We define the *significance* of the prime power $p_i^{e_i}$, $\operatorname{sig}(p_i^{e_i})$, to be

$$\operatorname{sig}(p_i^{e_i}) = \frac{1}{\sigma(p_i^{e_i}) + \frac{i-1}{k}}.$$

Thus, for two prime powers p^e and q^f, if $\sigma(p^e) < \sigma(q^f)$ then $\operatorname{sig}(p^e) > \operatorname{sig}(q^f)$, and in the event that $\sigma(p^e) = \sigma(q^f)$ and $p < q$, we have $\operatorname{sig}(p^e) > \operatorname{sig}(q^f)$.

We extend the definition of significance to all natural numbers as follows. For $n > 1$, we take $\operatorname{sig}(n) = \min\{\operatorname{sig}(p^e) : p^e \mid n\}$. Finally, if $n = 1$, we take $\operatorname{sig}(1) = 1$.

We now order the prime powers p^e by decreasing significance and construct primitive non-deficient numbers whose prime power factors have bounded significance. The sequence $P = (p_i^{e_i})_{i=1}^{\infty}$ of prime powers ordered by significance thus begins

$$2,\ 3,\ 5,\ 2^2,\ 7,\ 11,\ 3^2,\ 13,\ 2^3,\ 17,\ 19,\ 23,\ 29,\ 2^4,\ 5^2,\ 31,\ 37,\ 3^3,\ 41,\ 43,\ 47,\ 53,\ 7^2, \ldots.$$

Remark 5.11. Note that the ordering of prime powers by significance differs from the natural ordering in that prime powers p^e for $e > 1$ show up later than they would otherwise. Note also with the notation for prime powers ordered by significance, $P = (p_i^{e_i})_{i=1}^{\infty}$, p_i may be equal to p_j for $i \neq j$. In fact, each prime p is equal to p_i for infinitely many i's.

5.4 An ordering of α-pnd's

We are now in a position to construct a sequence of α-pnd's that satisfies the conditions of Theorem 5.9. We will order the primitive non-deficient numbers using the prime power sequence P in the following manner.

For each term $p_i^{e_i}$ in P, we consider the set of α-pnd's which have $p_i^{e_i}$ as the least significant prime power factor, which we will call the $p_i^{e_i}$*-block*. The sequence of blocks found in this way contains all α-pnd's since any α-pnd has a unique least significant prime power factor, and such a α-pnd will be found in the corresponding block. We will then say that $\mathbb{P}$ is ordered by significance. If we wish, we could further order the members of each block, say using lexicographic ordering by significance, but in fact we will not be concerned with how elements are ordered within each block.

To demonstrate this ordering, we will construct a list of the first few blocks for $\alpha = 2$. The list would begin by choosing $p_1^{e_1} = 2^1$. However, since 2 is deficient and there are no prime powers preceding it, there are no pnd's corresponding to this choice. Hence the 2^1-block is empty. We now move on to the next term, $p_2^{e_2} = 3^1$. The number 3 by itself is deficient, but $3 \cdot 2$ is a pnd, so we have found the first one on our list. Since we have exhausted our possibilities with 3^1, this completes the 3^1-block. The next term of P is 5^1, and we construct pnds that have 5 as a factor which also contain 2 or 3. Since we want to avoid multiples of 6, we need only check that neither 10 nor 15 are abundant. Next we use 2^2. We do not consider 2 since 2 divides 2^2, and we need not consider 3 since we have already counted 6. Thus, we check that $2^2 \cdot 5$ is abundant, and that it is primitive since removing either a 5 or a 2 makes the number deficient. In fact, due to our ordering we only need to check that removing a 2 makes the number deficient. The next term, 7, gives us two pnd's, $2 \cdot 5 \cdot 7$ and $2^2 \cdot 7$. Thus, there are two elements in the 7-block. Proceeding in this manner, we generate a list of blocks of pnd's

$$\{\}, \quad \{2 \cdot 3\}, \quad \{\}, \quad \{5 \cdot 2^2\}, \quad \{2 \cdot 5 \cdot 7, 2^2 \cdot 7\}, \quad \{\}, \quad \{5 \cdot 7 \cdot 11 \cdot 3^2\}, \ldots.$$

To state the next theorem, we introduce the notation

$$L_k := \operatorname{lcm}\{p_i^{e_i} : i \leq k\}. \tag{5.6}$$

Theorem 5.12. *The ordering of α-pnd's by significance satisfies the conditions of Theorem 5.9. In fact, the reduced cofactor set for each α-pnd a consists only of primes, and is given explicitly for a in block $p_k^{e_k}$ as*

$$C' = \{p : p \mid L_k/a\}.$$

Proof. It suffices to show that the primes p in the cofactor set C for a in block $p_k^{e_k}$ are exactly those satisfying $p \mid L_k/a$, and that each composite $c \in C$ is divisible by some prime $p \in C$.

First suppose $p \mid L_k/a$. Then $p \neq p_k$ and ap/p_k is abundant, since if $p^e \| a$, then p^{e+1} has greater significance than $p_k^{e_k}$. Thus, ap/p_k has an α-pnd divisor a' appearing before a, and

$$\frac{a'}{(a', a)} = p.$$

Thus, all the primes claimed are in C'.

Conversely, say a' appears before a in the sequence and $p \mid a'/(a', a)$. Then for some $e > 0$, $p^e \| a'$ and $p^e \nmid a$. But $p^e \mid L_k$. Thus, $p \mid L_k/a$.

So, we have shown that each prime dividing L_k/a is in C and that each prime factor of each $c \in C$ divides L_k/a. Thus, C' is the set of primes dividing L_k/a.

□

By this theorem, we now have a compact way of writing the product in (5.5):

$$\prod_{c \in C_i'} \left(1 - \frac{1}{c}\right) = \frac{\varphi(c_i)}{c_i}, \qquad c_i = \prod_{c \in C_i'} c. \tag{5.7}$$

Then we can extract the relevant information for each α-pnd a_i in the single number c_i. We define the sequence

$$\mathbb{C} = (c_i)_{i=1}^{\infty}$$

to be the cofactor sequence for the sequence of pnd's $\mathbb{P}$ ordered by significance, and in general let $\mathbb{C}_\alpha$ be the cofactor sequence for the sequence of α-pnd's ordered by significance. Thus, we have the following corollary.

Corollary 5.13. *Let P_1 be any subsequence of P ordered by significance such that if p^e is a term of P_1, then so is p^f for $1 \leq f < e$. Let $\mathbb{P}_1$ be the sequence of α-pnd's formed using P_1 and the procedure described in this section, with $\mathbb{P}_1 = (a_i)_{i=1}^r$ ordered by significance. Let $(c_i)_{i=1}^r$ be the cofactor sequence for $\mathbb{P}_1$. Then the density of the set of multiples of $\mathbb{P}_1$ is given by*

$$\mathbf{d}\,\mathscr{M}(\mathbb{P}_1) = \sum_{i=1}^{r} \frac{\phi(c_i)}{c_i} \cdot \frac{1}{a_i} = \sum_{i=1}^{r} \frac{\phi(L_k/a_i)}{L_k},$$

where k is the index of the block in which a_i belongs.

This sum allows us to calculate a lower bound for the density of abundant numbers and generalizes Behrend's calculation for a large class of subsets of α-pnd's. We will call this the α-pnd method, or simply the pnd method when $\alpha = 2$.

We conclude this section with a table of the first few pnd's ordered by significance, the block they belong to, their cofactor sequence, their reduced cofactor, and the corresponding reduced cofactor set.

a	block	cofactor sequence	L_k/a	reduced cofactor set
$2 \cdot 3$	3		1	
$5 \cdot 2^2$	2^2	(3)	3	$\{3\}$
$2 \cdot 5 \cdot 7$	7	$(3, 2)$	$3 \cdot 2$	$\{3, 2\}$
$2^2 \cdot 7$	7	$(3, 5, 5)$	$3 \cdot 5$	$\{3, 5\}$
$5 \cdot 7 \cdot 11 \cdot 3^2$	3^2	$(2, 2^2, 2, 2^2)$	2^2	$\{2\}$

5.5 Asymptotics of the pnd method when $\alpha = 2$

Let us denote by $\mathbb{P}[y]$ the set of pnd's in $\mathbb{P}$ consisting of pnd's from p^e-blocks with $\sigma(p^e) \leq y$. By the pnd method we can calculate the density $\mathbf{d}\mathscr{M}(\mathbb{P}[y])$. We now ask for a bound on the error $\mathbf{d}\mathscr{A} - \mathbf{d}\mathscr{M}(\mathbb{P}[y])$. A simple bound can be found by taking the reciprocal sum of the elements of the set $\mathbb{P} \setminus \mathbb{P}[y]$. We first show that this set is contained in $\mathbb{P} \setminus \mathbb{P}(y)$, so that

$$\sum_{\substack{a \in \mathbb{P} \\ a > y}} \frac{1}{a} \tag{5.8}$$

is an upper bound. The containment can be seen by the chain of implications

$$\sigma(p^e) > y \implies p^e > y/2 \implies 2p^e > y \implies a > y,$$

where the first implication is from the observation that for any prime power p^e we have $h(p^e) < 2$, and the final implication uses that p^e is a proper divisor of a. In fact, with a little more work we may improve the bound on a.

Lemma 5.14. *Let a be a pnd. If $p^e \| a$ and $\operatorname{sig}(a) = \operatorname{sig}(p^e)$, where $\sigma(p^e) > y$, then*

$$a > \max\left\{\frac{py}{2}, \frac{p^{e-1}(y-1)}{2}\right\}.$$

Independent of p, we have

$$a > \max\left\{\left(\frac{y}{2}\right)^{1+\frac{1}{e}}, \frac{y-1}{y}\left(\frac{y}{2}\right)^{2-\frac{1}{e}}\right\}.$$

Finally, independent of e, we have

$$a > \frac{y-1}{y}\left(\frac{y}{2}\right)^2.$$

Proof. We will factor the pnd a in two different ways to determine two bounds on a. First, write $a = a'p$. Since a is a pnd,

$$2 > h(a') \geq 2\frac{h(p^{e-1})}{h(p^e)} = 2\left(\frac{\sigma(p^{e-1})p^e}{p^{e-1}\sigma(p^e)}\right) = 2\left(\frac{\sigma(p^e)-1}{\sigma(p^e)}\right) = 2 - \frac{1}{\sigma(p^e)/2}.$$

We now use that a rational number in an interval $(0, 1/n]$ must have denominator d with $d \geq n$. Thus, writing $2 - h(a')$ as the fraction $(2a' - \sigma(a'))/a'$, we have $a' \geq \sigma(p^e)/2 > y/2$ so $a > py/2$. To remove the dependence on p, we determine a bound on p. Since $h(p^e) < p/(p-1) < 2$, we have $p^e > \sigma(p^e)/2 > y/2$. We conclude that

$$a = a'p > \left(\frac{y}{2}\right)^{1+1/e}.$$

We now factor a as $a = a''p^e$. Then we have

$$2\frac{p^{e-1}}{\sigma(p^{e-1})} > h(a'') \geq 2\frac{p^e}{\sigma(p^e)}.$$

Multiplying the inequalities through by $p/2$, we have

$$p\frac{p^{e-1}}{\sigma(p^{e-1})} > \frac{ph(a'')}{2} \geq p\frac{p^e}{\sigma(p^e)}.$$

Next we use that $p^e = \sigma(p^e) - \sigma(p^{e-1})$ so that

$$\frac{\sigma(p^e) - \sigma(p^{e-1})}{\sigma(p^{e-1})} > \frac{ph(a'')}{2} \geq \frac{\sigma(p^{e+1}) - \sigma(p^e)}{\sigma(p^e)}.$$

Finally, we use $\sigma(p^e) = p\sigma(p^{e-1}) + 1$ so the inequalities become

$$\frac{(p-1)\sigma(p^{e-1}) + 1}{\sigma(p^{e-1})} > \frac{ph(a'')}{2} \geq \frac{(p-1)\sigma(p^e) + 1}{\sigma(p^e)}.$$

Thus, we have that

$$\frac{1}{\sigma(p^{e-1})} > \frac{ph(a'')}{2} - (p-1) \geq \frac{1}{\sigma(p^e)} > 0,$$

which means that the fraction $ph(a'')/2 = p\sigma(a'')/2a''$ has denominator $2a''$ bounded by

$$2a'' > \sigma(p^{e-1}) = \frac{\sigma(p^e) - 1}{p} > \frac{y-1}{p}.$$

Thus, $a = a''p^e > p^{e-1}y/2$. We remove the dependence on p using $p^e > \sigma(p^e)/2 > y/2$ so that

$$a = a''p \cdot p^{e-1} > \frac{y-1}{2} \cdot \left(\frac{y}{2}\right)^{\frac{e-1}{e}} = \frac{y-1}{y}\left(\frac{y}{2}\right)^{2-\frac{1}{e}}.$$

The third assertion of the lemma follows from the second by separately considering the cases $e = 1$ and $e \geq 2$. □

Although this result is not significant asymptotically, it can be used to advantage for computational purposes.

For purposes of asymptotics we will return to the simpler bound $a > y$ for the remainder of this section. We now show that as $y \to \infty$, the error bound (5.8) goes to zero. By partial summation we can write

$$\sum_{a\in\mathbb{P},a>y} \frac{1}{a} = \left[\frac{|\mathbb{P}(n)|}{n}\right]_y^\infty + \int_y^\infty \frac{|\mathbb{P}(n)|}{n^2} dn.$$

From [12], we have the upper bound

$$|\mathbb{P}(n)| \leq n \exp\left(-\frac{1}{25}\sqrt{\log n \log\log n}\right)$$

for n larger than some $n_0(\epsilon)$. Since

$$\begin{aligned}\sum_{a\in\mathbb{P},a>y} \frac{1}{a} &\leq \int_y^\infty \frac{|\mathbb{P}(t)|}{t^2} dt \\ &\leq \int_y^\infty \frac{1}{te^{\frac{1}{25}\sqrt{\log t \log\log t}}} dt \\ &= O\left(\left(\frac{\log y}{\log\log y}\right)^{1/2} \exp\left(-\frac{1}{25}(\log y \log\log y)^{1/2}\right)\right)\end{aligned}$$

for sufficiently large y, we have that the error does indeed go to zero. We have thus proved the following theorem and corollary.

Theorem 5.15. *The error* $\mathbf{d}\mathscr{A} - \mathbf{d}\mathscr{M}(\mathbb{P}[y])$ *behaves as*

$$\mathbf{d}\mathscr{A} - \mathbf{d}\mathscr{M}(\mathbb{P}[y]) = O\left(\left(\frac{\log y}{\log\log y}\right)^{1/2} \exp\left(-\frac{1}{25}(\log y \log\log y)^{1/2}\right)\right)$$

for sufficiently large y.

Corollary 5.16. *The density of abundant numbers can be expressed as the infinite sum*

$$\mathbf{d}\mathscr{A} = \sum_{a_i\in\mathbb{P}} \frac{\varphi(c_i)}{c_i}\frac{1}{a_i},$$

where the a_i are pnd's and $c_i = L_k/a_i$, as defined in (5.7)*, and where L_k is as defined in* (5.6)*.*

5.6 Special values of α

Recall that the Deléglise method uses a different infinite sum expression for $\mathbf{d}\,\mathscr{A}$ which can be extended to determine in general the density of α-nondeficient numbers. We have shown that the error term goes to zero uniformly in α. In contrast, we cannot extend the error estimate for the pnd method to the case for α-pnd's. In fact, to prove Theorem 5.15 we have used the result in [12] which makes special use of the value $\alpha = 2$. In contrast, Erdős in [14] presents an example of α for which the sum of reciprocals of α-pnd's does not converge. We will begin by supplying a proof of this fact, which is only stated in the Erdős paper.

Proposition 5.17. *Let $p_1, p_2, \ldots$ be an infinite sequence of primes satisfying $p_{k+1} > \exp(\exp(p_k^2))$. Define α to be*

$$\alpha := \prod_{k=1}^{\infty}\left(1 + \frac{1}{p_k}\right) = \lim_{k\to\infty} \frac{\sigma(p_1p_2\cdots p_k)}{p_1p_2\cdots p_k}.$$

Then the sum of reciprocals of α-pnd's does not converge.

Proof. First we check that this infinite product converges. We note that since $\exp(x) > x$, we also have $\exp(\exp(x)) > \exp(x) > x$, which we use to show

$$p_k > \exp(\exp(p_{k-1}^2)) > p_{k-1}^2.$$

Then

$$\sum_{k=1}^{\infty} \frac{1}{p_k} < 1 + \sum_{k=2}^{\infty} \frac{1}{p_{k-1}^2} < \zeta(2),$$

establishing convergence. Finally,

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{p_k}\right) < \exp\left(\sum_{k=1}^{\infty} \log\left(1 + \frac{1}{p_k}\right)\right) < \exp\left(\sum_{k=1}^{\infty} \frac{1}{p_k}\right) < \exp(\zeta(2)),$$

so the infinite product also converges.

Next we verify that the numbers $n = p_1 p_2 \cdots p_k p$, for primes $p_k < p < p_{k+1}$, are all α-pnd's. To see that n is α-nondeficient, note that

$$h(n) = \prod_{i=1}^{k} \left(1 + \frac{1}{p_i}\right) \cdot \left(1 + \frac{1}{p}\right),$$

so $h(n) \geq \alpha$ follows if

$$1 + \frac{1}{p} \geq \prod_{i=k+1}^{\infty} \left(1 + \frac{1}{p_i}\right).$$

First we establish a lower bound for the left side of this inequality. Since $p \leq p_{k+1} - 2$,

$$1 + \frac{1}{p} \geq 1 + \frac{1}{p_{k+1} - 2} = \left(1 + \frac{1}{p_{k+1}}\right) \left(\frac{1 + \frac{1}{p_{k+1}-2}}{1 + \frac{1}{p_{k+1}}}\right).$$

The final factor is

$$\frac{p_{k+1}(p_{k+1} - 2) + p_{k+1}}{(p_{k+1} + 1)(p_{k+1} - 2)} = 1 + \frac{2}{(p_{k+1} + 1)(p_{k+1} - 2)} > 1 + \frac{2}{p_{k+1}^2 - 1} > 1 + \frac{2}{p_{k+1}^2}.$$

We continue the estimate by using the definition of p_{k+2} to write

$$1 + \frac{2}{p_{k+1}^2} > 1 + \frac{2}{\log\log p_{k+2}}.$$

Thus, we see that we have reduced our problem to establishing

$$1+\frac{2}{\log\log p_{k+2}} \geq \prod_{i=k+2}^{\infty}\left(1+\frac{1}{p_i}\right).$$

Now we turn our attention to the right side of this inequality and determine an upper bound that is easy to compare to the left side. We first use the bound

$$\prod_{i=k+2}^{\infty}\left(1+\frac{1}{p_i}\right)=\exp\left(\sum_{i=k+2}^{\infty}\log\left(1+\frac{1}{p_i}\right)\right)<\exp\left(\sum_{i=k+2}^{\infty}\frac{1}{p_i}\right).$$

For the final infinite sum, we iterate the inequality established earlier of $p_k > p_{k-1}^2$ to get

$$p_{k+i} > p_{k-1}^{2^i}.$$

Thus,

$$\sum_{i=k+2}^{\infty}\frac{1}{p_i}<\frac{1}{p_{k+2}}+\sum_{i=1}^{\infty}\frac{1}{p_{k+2}^{2^i}}<\frac{2}{p_{k+2}},$$

and since $\exp(x) < 1+2x$ for $0 < x \leq 1$,

$$\exp\left(\sum_{i=k+2}^{\infty}\frac{1}{p_i}\right)<\exp\left(\frac{2}{p_{k+2}}\right)<1+\frac{4}{p_{k+2}}.$$

It remains to demonstrate that

$$\log\log x < \frac{x}{2}$$

for $x > 1$. This can be seen via the inequalities $e^x > x > \log x$ for $x > 0$ and by noting that $x - \log x$ has minimum value 1 for $x > 0$. Then

$$e^x > x \geq \log x + 1 > \log x + \log 2 = \log 2x,$$

so taking logs and replacing x by $x/2$ gives us what we wanted to show.

Since we have identified a subset of the primitive α-nondeficient numbers, it remains to show that the reciprocal sum of these numbers diverges. We first factor the sum according to

$$\sum_{k=1}^{\infty} \frac{1}{p_1 \cdots p_{k+1}} \sum_{p_k < p < p_{k+1}} \frac{1}{p}$$

and bound the inner sum. Using

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + O(1) \qquad \text{and} \qquad \log\log x < \frac{x}{2} \leq \frac{x^2}{2}$$

for $x \geq 1$, along with the definition of p_{k+1}, we get

$$\begin{aligned} \sum_{p_k < p < p_{k+1}} \frac{1}{p} &= \log\log p_{k+1} - \log\log p_k + O(1) \\ &\geq p_k^2 - \frac{1}{2} p_k^2 + O(1) \\ &= \frac{1}{2} p_k^2 + O(1). \end{aligned}$$

Next, we prove by induction that $p_k > p_1 p_2 \cdots p_{k-1}$. For $k = 2$, this is clear by definition. Assuming the validity of case $k-1$, we have $p_1 p_2 \cdots p_{k-1} < p_{k-1}^2 < p_k$, as claimed. We use this result to show divergence of our sum:

$$\begin{aligned} \sum_{k=1}^{K} \frac{1}{p_1 \cdots p_{k+1}} \sum_{p_k < p < p_{k+1}} \frac{1}{p} &\geq \sum_{k=1}^{K} \frac{1}{p_k^2} \cdot \left(\frac{1}{2} p_k^2 + O(1) \right) \\ &= \frac{1}{2} K + O(1). \end{aligned}$$

Since the sum diverges as $K \to \infty$, we have proven our result. □

5.7 Liouville numbers

A number δ is called a *Liouville number* if for each k there exists a rational number a_k/b_k, with $a_k, b_k \in \mathbb{Z}$, such that

$$0 < \left|\delta - \frac{a_k}{b_k}\right| < \frac{1}{b_k^k}.$$

This condition implies that δ is irrational, and moreover that it is transcendental. In [14], Erdős stated that the α defined in Proposition 5.17 is a Liouville number, and in general that if for some value α the primitive α-abundant numbers have divergent reciprocal sum, then α must be Liouville. We first provide a proof of the first claim.

Proposition 5.18. *The number α defined in Proposition 5.17 is Liouville.*

Proof. Define

$$\frac{a_k}{b_k} = \prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right), \qquad b_k = \prod_{i=1}^{k} p_i.$$

Then it suffices to prove that the sequence of rational numbers a_k/b_k approximates α better than $1/b_k^k$ for all k. Thus, we need to prove that

$$\left|\alpha - \frac{a_k}{b_k}\right| = \prod_{i=1}^{\infty}\left(1 + \frac{1}{p_i}\right) - \prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right) < \frac{1}{b_k^k},$$

or, dividing both sides by a_k/b_k,

$$\prod_{i=k+1}^{\infty}\left(1 + \frac{1}{p_i}\right) - 1 < \frac{1}{b_k^k \prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right)}.$$

We will proceed by bounding above the left side of this inequality:

$$\begin{aligned}\prod_{i=k+1}^{\infty}\left(1+\frac{1}{p_i}\right)-1 &< \exp\left(\sum_{i=k+1}^{\infty}\frac{1}{p_i}\right)-1\\ &< \frac{3}{2}\sum_{i=k+1}^{\infty}\frac{1}{p_i}\\ &= \frac{3}{2}\frac{1}{p_{k+1}}+\frac{3}{2}\frac{1}{p_{k+2}}+\frac{3}{2}\sum_{i=k+3}^{\infty}\frac{1}{p_i}\\ &< \frac{3}{2}\frac{1}{p_{k+1}}+\frac{3}{2}\frac{1}{p_{k+2}}+\frac{3}{2}\sum_{i=k+2}^{\infty}\frac{1}{p_i^2}\\ &< \frac{3}{2}\frac{1}{p_{k+1}}+\frac{3}{2}\frac{1}{p_{k+1}^2}+\frac{3}{2}\frac{1}{p_{k+1}^2}\\ &< \frac{2}{p_{k+1}}.\end{aligned}$$

Next we prove by induction that

$$\left(2\prod_{i=1}^{k}p_i\right)^k < p_{k+1},$$

so that

$$\frac{2}{p_{k+1}} < \frac{2}{(2b_k)^k}.$$

From the definition for p_k we have $2p_1 < p_2$. Now assuming the case $k-1$, and using the result $p_k > \prod_{i=1}^{k-1} p_i$, we have

$$\left(2\prod_{i=1}^{k-1}p_i\right)^{k-1}\cdot\prod_{i=1}^{k-1}p_i\cdot p_k^k < \left(2\prod_{i=1}^{k}p_i\right)^k < 2^k p_k^{k+2}.$$

Thus, it remains to show

$$2^k p_k^{k+2} < e^{e^{p_k^2}},$$

since the expression on the right bounds p_{k+1} below by definition. Since $x+1 < x^2/2$ for $x > 3$ (and noting that $\exp\exp 4 \approx 5 \times 10^{23}$ so $p_k > 3$), we have

$$p_k + 2 < 1 + \frac{1}{2}p_k^2 < e^{\frac{1}{2}p_k^2},$$

and thus

$$\log\log(2^k p_k^{k+2}) < \log(p_k + 2) + \log\log 2p_k < 2\log(p_k + 2) < p_k^2.$$

Thus,

$$\prod_{i=k+1}^{\infty}\left(1 + \frac{1}{p_i}\right) - 1 < \frac{2}{p_{k+1}} < \frac{2}{(2b_k)^k} < \frac{1}{b_k \prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right)}$$

for $k > 1$. This proves that α is Liouville. □

Let $N_\alpha(n)$ denote the number of primitive α-abundant numbers in $[1, n]$. Erdős in [14] states that if α is not a Liouville number, then it can be shown using a proof similar to that found in [12] that

$$N_\alpha(n) < \frac{n}{e^{c_\alpha(\log n \log\log n)^{1/2}}}$$

for some positive constant c_α. Note that if α is non-Liouville, there must be some positive real number κ such that

$$\left|\alpha - \frac{a}{b}\right| > \frac{1}{b^\kappa}$$

for any $a, b \in \mathbb{Z}$ such that $\alpha \neq a/b$. Note also that if the inequality is satisfied for some $\kappa = \kappa_0$, then any value of κ greater than κ_0 will satisfy the inequality as well.

Thus we may assume that $\kappa \geq 2$.

We will prove the Erdős statement in terms of such an exponent κ. In fact we will establish explicit constants which is useful in an implementation of the upper bound α-pnd calculation for α non-Liouville. The proof of Erdős in [12] has subsequently been refined by Aleksandar Ivić [24] and Michael Avidon [2]. However, these refinements rely on the use of the counting function for y-smooth numbers $n \leq x$, $\Psi(x, y)$, which is difficult to make explicit. We have chosen instead to follow the earlier Erdős proof.

In the proof we will be factoring numbers n according to the power of their prime factors, as follows.

Definition 5.19. Let the *squarefree* part of n be the product of the prime factors of n that occur to the first power, and the *squarefull* part the product of the prime factors that occur to higher powers. For instance, the squarefree part of $2 \cdot 3 \cdot 5^2 \cdot 7^3$ is $2 \cdot 3$ and the squarefull part $5^2 \cdot 7^3$.

Theorem 5.20. *Let $\mathbb{P}_\alpha(x)$ denote the set of α-pnd's in $[1, x]$. Suppose α is a non-Liouville number and $\kappa \geq 2$ is an integer such that for any $a, b \in \mathbb{Z}$, $\alpha \neq a/b$,*

$$\left|\alpha - \frac{a}{b}\right| > \frac{1}{b^\kappa}.$$

Then

$$|\mathbb{P}_\alpha(x)| \leq \delta \frac{x}{e^{\frac{\beta}{12\kappa}(\log x \log\log x)^{1/2}}},$$

where

$$\beta = \frac{\sqrt{e} - \frac{1}{2}}{\frac{1}{6} + \sqrt{e}} = 0.632769033\ldots,$$

$$\delta = 8\sqrt[4]{\alpha}\frac{\zeta(3/2)}{\zeta(3)} = \sqrt[4]{\alpha} \cdot 17.3860345\ldots,$$

and x is large enough that

$$x \geq \exp(13100)$$

and

$$\kappa \leq \beta \frac{\sqrt{\log x \log\log x}}{36 \log \frac{3\zeta(3)}{\zeta(3/2)} + 3\log(\eta \log\log x)},$$

where

$$\eta = e^{\gamma} + \frac{5}{2(\log\log 223092871)^2} = 2.0671\ldots.$$

Proof. In what follows, we will use c_i to denote constants local to each lemma, and C_i to denote global constants. We define the functions

$$E_1 = E_1(x) := (\log x \log\log x)^{1/2}$$

and

$$E_2 = E_2(x) := \frac{E_1(x)}{\log\log x} = \left(\frac{\log x}{\log\log x}\right)^{1/2},$$

so with this notation our goal will be to prove that

$$|\mathbb{P}_\alpha(x)| < \delta \frac{x}{e^{\frac{\beta}{12\kappa}E_1}}$$

for some constants $\delta, \beta > 0$.

We first show that we can restrict our attention to numbers satisfying both of the following conditions for some constants $0 < C_1 < \frac{C_2}{4\kappa}$:

(A) if $n \leq x$, the squarefull part of n is less than $\frac{1}{\sqrt{\alpha}} e^{C_1 E_1}$,

(B) if $n \leq x$, the greatest prime factor of n is greater than $e^{C_2 E_1}$.

We first study the numbers not satisfying property (A). We will use the explicit bounds for the number of squarefull numbers up to x given in Golomb [19].

Lemma 5.21. *Let $U(x)$ denote the number of squarefull $n \le x$. Then*

$$C\sqrt{x} - 3\sqrt[3]{x} \le U(x) \le C\sqrt{x},$$

where $C = \frac{\zeta(3/2)}{\zeta(3)} = 2.173\ldots$.

Corollary 5.22. *The reciprocal sum of the squarefull numbers $n > x$ is*

$$\sum_{\substack{n>x \\ n \text{ squarefull}}} \frac{1}{n} = \frac{C}{\sqrt{x}} + E(x)$$

where

$$-\frac{9}{2}\frac{1}{x^{2/3}} \le E(x) \le \frac{3}{x^{2/3}}.$$

Proof. From Lemma 5.21 we have $U(x) = C\sqrt{x} + E_0(x)$ where

$$-3\sqrt[3]{x} \le E_0(x) \le 0.$$

Then by partial summation we have

$$\begin{aligned}
\sum_{\substack{n>x \\ n \text{ squarefull}}} \frac{1}{n} &= \int_x^\infty \frac{dU(t)}{t} \\
&= \left.\frac{U(t)}{t}\right|_x^\infty + \int_x^\infty \frac{U(t)}{t^2} dt \\
&= -\frac{C}{\sqrt{x}} - \frac{E_0(x)}{x} + \left.\frac{-2C}{\sqrt{t}}\right|_x^\infty + \int_x^\infty \frac{E_0(t)}{t^2} dt \\
&= \frac{C}{\sqrt{x}} + E(x),
\end{aligned}$$

where

$$E(x) := -\frac{E_0(x)}{x} + \int_x^\infty \frac{E_0(t)}{t^2}dt.$$

Then since

$$0 \leq -\frac{E_0(x)}{x} \leq \frac{3}{x^{2/3}}$$

and

$$-\frac{9}{2}\frac{1}{x^{2/3}} = -3\int_x^\infty \frac{1}{t^{5/3}}dt \leq \int_x^\infty \frac{E_0(t)}{t^2}dt \leq 0,$$

we have

$$-\frac{9}{2}\frac{1}{x^{2/3}} \leq E(x) \leq \frac{3}{x^{2/3}},$$

as asserted. □

Lemma 5.23. *Let the function f be nondecreasing on $x \geq x_0$ for some bound x_0. The number of $n \leq x$ with squarefull part not less than $f(x)$ is at most*

$$\left(\frac{C}{\sqrt{f(x)}} + \frac{3}{\sqrt[3]{f(x)}^2}\right)x.$$

Proof. The number of integers n up to x with squarefull part r not less than $f(x)$ is bounded by

$$\sum_{\substack{r>f(x)\\ r \text{ squarefull}}} \frac{x}{r},$$

so by Corollary 5.22, we have the upper bound

$$\left(\frac{C}{\sqrt{f(x)}} + \frac{3}{\sqrt[3]{f(x)}^2}\right)x,$$

proving our assertion. □

Thus, we have that the number of $n \leq x$ not satisfying condition (A) is bounded by

$$\left(\frac{C\sqrt[4]{\alpha}}{e^{\frac{C_1}{2}E_1}} + \frac{3\sqrt[3]{\alpha}}{e^{\frac{2C_1}{3}E_1}} \right) x.$$

We have

$$\frac{C\sqrt[4]{\alpha}}{e^{\frac{C_1}{2}E_1}} \geq \frac{3\sqrt[3]{\alpha}}{e^{\frac{2C_1}{3}E_1}}$$

when

$$e^{\frac{C_1}{6}E_1} \geq \frac{3}{C}\alpha^{\frac{1}{12}},$$

namely when

$$\sqrt{\log x \log\log x} \geq \frac{6}{C_1} \log \frac{3\sqrt[12]{\alpha}}{C}.$$

We conclude that the number of $n \leq x$ not satisfying condition (A) is bounded by

$$2C\sqrt[4]{\alpha}\frac{x}{e^{\frac{C_1}{2}E_1}} \qquad \text{for } x \text{ such that} \qquad \sqrt{\log x \log\log x} \geq \frac{6}{C_1} \log \frac{3\sqrt[12]{\alpha}}{C}.$$

To prove the result in connection with condition (B), we will use the following lemmas.

Lemma 5.24. *Let $a, b > 0$. Then for $x > e$, the minimum value of*

$$\frac{(\log x)^a}{(\log\log x)^b}$$

is

$$\left(\frac{ae}{b}\right)^b$$

and it occurs at

$$x = e^{e^{\frac{b}{a}}}.$$

Proof. We consider the derivative

$$\frac{d}{dx}\left(\frac{(\log x)^a}{(\log\log x)^b}\right) = \frac{(\log x)^{a-1}}{x(\log\log x)^b}\left(a - \frac{b}{\log\log x}\right),$$

and note that the function has a minimum and value as given. □

Now we will prove the statement in connection with condition (B).

Lemma 5.25. *Let $c_1, c_2, c_3 > 0$ be constants with*

$$c_1 < \sqrt{e} - 1,$$

$$c_2 > \frac{1}{2\sqrt{e}},$$

$$c_1 + \sqrt{e}c_2 < \sqrt{e} - \frac{1}{2},$$

$$\sqrt{e} - \frac{eC_3}{2} \le c_1 + C_3 e^{3/2} c_2$$

where $C_3 = 1.15993801$, and

$$c_3 = \frac{2(\sqrt{e} - c_1)}{1 + 2\sqrt{e}c_2}.$$

The number of integers $n \le x$ with squarefull part less than $e^{c_1 E_1}$ and greatest prime factor not greater than $e^{c_2 E_1}$ is less than

$$2x\left(\frac{eC_3 \log\log x}{c_3 E_2}\right)^{c_3 E_2}$$

for $x \ge 286$.

Proof. We divide the numbers under consideration into two classes. In the first class we place integers for which the number of different prime factors is less than or equal to c_3E_2 for some $c_3 > 0$. Since these also have squarefull part not greater than $e^{c_1E_1}$, the number of these at most x is less than or equal to the largest such number. An upper bound for such a number is

$$(e^{c_2E_1})^{c_3E_2} \cdot e^{c_1E_1} = e^{c_2c_3 \log x + c_1E_1} = \frac{x}{e^{(1-c_2c_3)\log x - c_1E_1}}.$$

We note that we will need $c_2c_3 < 1$.

We now consider the second class consisting of integers not greater than x where the number of different prime factors is greater than c_3E_2. Since such integers are all multiples of integers containing at least $s = \lfloor c_3E_2 \rfloor$ distinct prime factors, we can let $a_1, a_2, \ldots, a_t$ be the integers at most x which contain exactly s distinct prime factors and bound the number of integers in the second class by

$$\sum_{i=1}^{t} \frac{x}{a_i}.$$

By the multinomial theorem we can write

$$\sum_{i=1}^{t} \frac{1}{a_i} \leq \frac{1}{s!}\left(\sum_{p\leq x} \frac{1}{p}\right)^s.$$

We explicitly bound this sum using the following lemmas.

To bound the reciprocal sum of primes we use the bound of Rosser and Schoenfeld [29]

$$\sum_{p\leq x} \frac{1}{p} < \log\log x + B + \frac{1}{2\log^2 x}, \qquad x \geq 286,$$

where

$$B = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) = 0.2614972\ldots.$$

For convenience we replace this bound by $C_3 \log\log x$, where C_3 is a constant chosen to satisfy

$$C_3 \log\log x > \log\log x + B + \frac{1}{2\log^2 x}.$$

Since C_3 must be greater than

$$1 + \frac{B}{\log\log 286} + \frac{1}{2\log^2 286 \log\log 286} = 1.15993800\ldots,$$

we choose $C_3 = 1.15993801$. Then

$$\sum_{i=1}^{t} \frac{1}{a_i} < \frac{(C_3 \log\log x)^s}{s!}$$

for $x \geq 286$. To bound $s!$, we will use Inequality (3.10). Using these bounds and that $s = \lfloor c_3 E_2 \rfloor$, we continue with the bound

$$\sum_{i=1}^{t} \frac{1}{a_i} < \frac{1}{e}\left(\frac{eC_3 \log\log x}{c_3 E_2}\right)^{c_3 E_2} = \frac{1}{e}\left(\frac{eC_3}{c_3}\sqrt{\frac{(\log\log x)^3}{\log x}}\right)^{c_3 E_2}.$$

We now show that for appropriately chosen constants,

$$\frac{1}{e^{(1-c_2 c_3)\log x - c_1 E_1}} \leq \left(\frac{eC_3}{c_3}\sqrt{\frac{(\log\log x)^3}{\log x}}\right)^{c_3 E_2},$$

so that the second bound is larger than the first. We begin by taking logs, yielding

$$c_1E_1 - (1 - c_2c_3)\log x \le c_3E_2\left(\log\left(\frac{eC_3}{c_3}\right) + \frac{3}{2}\log\log\log x - \frac{1}{2}\log\log x\right).$$

Dividing both sides by E_2 gives

$$c_1\log\log x - (1 - c_2c_3)E_1 \le c_3\log\left(\frac{eC_3}{c_3}\right) + \frac{3}{2}c_3\log\log\log x - \frac{c_3}{2}\log\log x.$$

Thus, we need the inequality

$$\left(c_1 + \frac{c_3}{2}\right)\log\log x \le (1 - c_2c_3)E_1 + c_3\log\left(\frac{eC_3}{c_3}\right) + \frac{3}{2}c_3\log\log\log x$$

to be satisfied. Assuming $c_3\log(eC_3/c_3) \ge 0$, it will be enough to require that the inequality

$$\left(c_1 + \frac{c_3}{2}\right)\log\log x \le (1 - c_2c_3)E_1 \tag{5.9}$$

be satisfied. Note that the assumption is satisfied if $0 \le c_3 \le eC_3$. The inequality (5.9) can be written

$$\frac{2c_1 + c_3}{2 - 2c_2c_3} \le \sqrt{\frac{\log x}{\log\log x}},$$

and as the right side has minimum value $\sqrt{e}$, we need

$$\frac{2c_1 + c_3}{2 - 2c_2c_3} \le \sqrt{e},$$

or

$$c_3 \le \frac{2(\sqrt{e} - c_1)}{1 + 2\sqrt{e}c_2}.$$

Note that this bound is smaller than the earlier bound $c_3 < 1/c_2$. Note also that we must have $c_1 < \sqrt{e}$. We have thus found that the two classes of numbers under consideration are bounded by

$$2\left(\frac{eC_3 \log\log x}{c_3 E_2}\right)^{c_3 E_2}, \tag{5.10}$$

provided we choose

$$0 < c_3 \leq \min\left\{eC_3, \frac{2(\sqrt{e}-c_1)}{1+2\sqrt{e}c_2}\right\}$$

and

$$c_1 < \sqrt{e}.$$

We further impose the condition

$$\frac{2(\sqrt{e}-c_1)}{1+2\sqrt{e}c_2} \leq eC_3,$$

so that

$$\sqrt{e} - \frac{eC_3}{2} \leq c_1 + C_3 e^{3/2} c_2.$$

Since $eC_3 = 3.1530\ldots$ and $2\sqrt{e} = 3.297\ldots$, the expression on the left side of the inequality is positive. With this condition, our bounds on c_3 become

$$0 < c_3 \leq \frac{2(\sqrt{e}-c_1)}{1+2\sqrt{e}c_2}.$$

Since a larger choice of c_3 gives a smaller bound (5.10), we choose

$$c_3 = \frac{2(\sqrt{e}-c_1)}{1+2\sqrt{e}c_2}.$$

Next, we find conditions satisfying $c_3 > 0$. Solving for

$$\frac{2(\sqrt{e} - c_1)}{1 + 2\sqrt{e}c_2} > 0,$$

we find

$$c_1 + \sqrt{e}c_2 < \sqrt{e} - \frac{1}{2}.$$

We now check when the two inequalities

$$\sqrt{e} - \frac{eC_3}{2} \leq c_1 + C_3 e^{3/2} c_2 \tag{5.11}$$

and

$$c_1 + \sqrt{e}c_2 < \sqrt{e} - \frac{1}{2} \tag{5.12}$$

have solutions c_1, c_2. Solving for c_1, we have

$$\sqrt{e} - \frac{eC_3}{2} - C_3 e^{3/2} c_2 \leq c_1 < \sqrt{e} - \frac{1}{2} - \sqrt{e}c_2.$$

For this to be a nonempty interval, we need

$$\sqrt{e} - \frac{eC_3}{2} - C_3 e^{3/2} c_2 < \sqrt{e} - \frac{1}{2} - \sqrt{e}c_2.$$

Solving for c_2, we have $c_2 > 1/(2\sqrt{e})$.

Solving the pair of inequalities (5.11) and (5.12) for c_2, we have

$$\frac{\sqrt{e} - \frac{eC_3}{2} - c_1}{C_3 e^{3/2}} \leq c_2 < \frac{\sqrt{e} - \frac{1}{2} - c_1}{\sqrt{e}}.$$

this time we need

$$\frac{\sqrt{e} - \frac{eC_3}{2} - c_1}{C_3 e^{3/2}} < \frac{\sqrt{e} - \frac{1}{2} - c_1}{\sqrt{e}},$$

which means $c_1 < \sqrt{e}$, which we already have. In fact, we must also consider our new bound, $c_2 > 1/(2\sqrt{e})$. Thus, we must have

$$\frac{1}{2\sqrt{e}} < \frac{\sqrt{e} - \frac{1}{2} - c_1}{\sqrt{e}}.$$

Solving for c_1, we now have $c_1 < \sqrt{e} - 1$. Likewise this bound on c_1 must be compatible with the lower bound for c_1, so

$$\sqrt{e} - \frac{eC_3}{2} - C_3 e^{3/2} c_2 < \sqrt{e} - 1.$$

Thus

$$\frac{1 - \frac{eC_3}{2}}{C_3 e^{3/2}} < c_2.$$

However, we already have $0 < c_2$, so this does not give an additional constraint. This establishes the lemma.

□

We now compare the bounds considered thus far.

Lemma 5.26. *The number of $n \leq x$ not satisfying properties (A) and (B) is bounded by*

$$4\sqrt[4]{\alpha} C \frac{x}{e^{\frac{C_1}{2} E_1}},$$

where $C_1 > 0$ and x satisfy the following contraints: Let C_2, C_4 be constants with

$$C_4 = \frac{2(\sqrt{e} - C_1)}{1 + 2\sqrt{e}C_2}$$

such that $C_4 > C_1$,

$$C_1 < \sqrt{e} - 1,$$

$$C_2 > \frac{1}{2\sqrt{e}}.$$

In addition

$$\sqrt{e} - \frac{eC_3}{2} \le C_1 + e^{3/2}C_2C_3$$

and

$$C_1 + \sqrt{e}C_2 < \sqrt{e} - \frac{1}{2}.$$

Let x be sufficiently large that

$$\frac{\log\log x}{\log\log\log x} \ge \frac{4C_4}{C_4 - C_1},$$

$$\sqrt{\log x \log\log x} \ge \frac{6}{C_1} \log \frac{3\sqrt[12]{\alpha}}{C},$$

and $x \ge \exp\exp((eC_3/C_4)^2)$, where $C_3 = 1.15993801$ as defined in Lemma 5.25.

Proof. Since $2\sqrt[4]{\alpha}C > 1$, we apply Lemma 5.25 directly and then use the bound

$$2\sqrt[4]{\alpha}C\frac{x}{e^{\frac{C_1}{2}E_1}} > \frac{x}{e^{\frac{C_1}{2}E_1}}.$$

Thus we first consider when the inequality

$$\frac{1}{e^{\frac{C_1}{2}E_1}} \geq \left(\frac{eC_3 \log\log x}{C_4E_2}\right)^{C_4E_2}$$

is satisfied. We proceed as in Lemma 5.25 and begin by taking logs. Then we have

$$-\frac{C_1}{2}E_1 \geq C_4E_2\left(\log\frac{eC_3}{C_4} + \frac{3}{2}\log\log\log x - \frac{1}{2}\log\log x\right).$$

Next we divide by C_4E_2 to get

$$-\frac{C_1}{2C_4}\log\log x \geq \log\frac{eC_3}{C_4} + \frac{3}{2}\log\log\log x - \frac{1}{2}\log\log x.$$

Rearranging so that all the terms are positive, we have

$$\frac{1}{2}\left(1 - \frac{C_1}{C_4}\right)\log\log x \geq \log\frac{eC_3}{C_4} + \frac{3}{2}\log\log\log x,$$

provided $C_1 < C_4$. To simplify this bound, we replace the expression $\log(eC_3/C_4)$ by $(1/2)\log\log\log x$, which we may do when

$$x \geq \exp\exp\left(\left(\frac{eC_3}{C_4}\right)^2\right).$$

Thus, we are left to satisfy the simpler inequality

$$\frac{1}{2}\left(1 - \frac{C_1}{C_4}\right)\log\log x \geq 2\log\log\log x.$$

Thus, we have proved the lemma. □

Before continuing, we state the following lemmas which we will use in the sequel.

Lemma 5.27. *For natural numbers* $n \neq 2$ *or* 6*, every* $m \leq n$ *has at most* $\log n$ *distinct prime factors.*

Proof. We first check when $\omega(m) \leq \log m$. If $\omega(m) = 0$, then $m = 1$, which has $\log 1 = 0$ distinct prime factors. If $\omega(m) = 1$, then all $m \geq e$ has $\log m \geq 1$ so 2 is the only exception in this case. If $\omega(m) \geq 2$, we have $m \geq 6 \cdot 5^{\omega(m)-2}$. This last inequality holds if and only if

$$\log m \geq \log 6 + (\omega(m) - 2) \log 5.$$

Solving for $\omega(m)$, we have

$$\omega(m) \leq \frac{\log m - \log 6}{\log 5} + 2.$$

Thus, we find $\omega(m) \leq \log m$ when

$$\frac{\log m - \log 6}{\log 5} + 2 \leq \log m.$$

Solving for $\log m$ gives

$$\log m \geq \frac{2 \log 5 - \log 6}{\log 5 - 1},$$

so we want

$$m \geq (25/6)^{1/(\log 5 - 1)} = 10.398\ldots.$$

For $m \leq 10$ we check that the only values m such that $\omega(m) > \log m$ is when $m = 2, 6$. □

In order to bound the size of α, we will make use of an explicit version of the following standard result which can be found, for instance, as Theorem 323 in [22].

Lemma 5.28. *Let* $N = \prod_{i=1}^{9} p_i = 223092870$. *For natural numbers* $n > N = \exp(19.22\ldots)$, *we have* $h(n) \leq C_6 \log\log n$, *where*

$$C_6 = e^\gamma + \frac{5}{2(\log\log(N+1))^2} = 2.06715\ldots.$$

Proof. We first observe that

$$h(n) = \frac{\sigma(n)}{n} < \frac{n}{\varphi(n)}.$$

Then by the Rosser and Schoenfeld [29] bound

$$\frac{n}{\varphi(n)} < e^\gamma \log\log n + \frac{5}{2\log\log n}$$

for $n > N$, we have for some constant C_6 that

$$\frac{n}{\varphi(n)} < C_6 \log\log n.$$

To determine C_6, we set

$$e^\gamma \log\log n + \frac{5}{2\log\log n} \leq C_6 \log\log n$$

and solve for C_6. Since $n > N$, we may take

$$C_6 = e^\gamma + \frac{5}{2(\log\log(N+1))^2} = 2.06715\ldots.$$

This gives us our result. □

It follows from this lemma that we need only consider α such that $1 < \alpha < C_6 \log\log n$.

Thus far our argument has not depended on our numbers being primitive α-nondeficient. We now show that we can further restrict the set of α-pnd's left to consider so that, in addition to (A) and (B), they satisfy the following:

(C) The squarefree part of each such α-pnd has a divisor d with $\sqrt{\alpha} e^{\frac{C_2}{4\kappa} - C_1} E_1 < d \le \frac{1}{\sqrt{\alpha}} e^{\frac{C_2}{2\kappa} E_1}$.

(D) If $a \le x$ is an α-pnd satisfying (A) and (B), then

$$\alpha \le h(a) < \alpha + \frac{\alpha}{e^{C_2 E_1}}.$$

To prove the statement regarding (C) we will use the following lemma.

Lemma 5.29. *Let $\alpha > 1$ and $C_1 \le C_2/(4\kappa)$. An α-pnd a with $6 < a \le x$ satisfying (A) and (B) has a divisor d such that $e^{\frac{C_2}{4\kappa} E_1} < d \le \frac{1}{\sqrt{\alpha}} e^{\frac{C_2}{2\kappa} E_1}$ when x is sufficiently large that*

$$\frac{3}{2} \log \alpha + \log\log x \le \frac{C_2}{4} \sqrt{\log x \log\log x}.$$

Proof. Let a be an α-pnd with $6 < a \le x$ satisfying (A) and (B). If a contains a prime factor in the desired interval we are done, so assume not and write $a = uv$, where u contains only prime factors not greater than $\frac{1}{\sqrt{\alpha}} e^{\frac{C_2}{4\kappa} E_1}$ and v contains only prime factors greater than $\frac{1}{\sqrt{\alpha}} e^{\frac{C_2}{2\kappa} E_1}$. Note that by (B), $v \ne 1$. Now we show that for x sufficiently large, $u > e^{\frac{C_2}{4\kappa} E_1}$. Suppose not. Since $h(u) < \alpha$, we have by the definition of κ that

$$\alpha - h(u) = \alpha - \frac{\sigma(u)}{u} > \frac{1}{u^{\kappa}}.$$

Then

$$h(u) < \alpha - \frac{1}{u^{\kappa}} \leq \alpha - \frac{1}{e^{\frac{C_2}{4}E_1}}.$$

Further,

$$h(v) = \prod_{p|v} \left(1 + \frac{1}{p}\right),$$

since by (A) the prime factors of v occur only to the first power. Hence by Lemma 5.27, for $x > 6$

$$h(v) < \left(1 + \frac{\sqrt{\alpha}}{e^{\frac{C_2}{2}E_1}}\right)^{\log x} < \exp\left(\frac{\sqrt{\alpha}\log x}{e^{\frac{C_2}{2}E_1}}\right).$$

Consequently,

$$\begin{aligned} h(a) &= h(u)h(v) \\ &< \exp\left(\log \alpha + \log\left(1 - \frac{1}{\alpha e^{\frac{C_2}{4}E_1}}\right) + \frac{\sqrt{\alpha}\log x}{e^{\frac{C_2}{2}E_1}}\right) \\ &< \alpha \exp\left(-\frac{1}{\alpha e^{\frac{C_2}{4}E_1}} + \frac{\sqrt{\alpha}\log x}{e^{\frac{C_2}{2}E_1}}\right). \end{aligned}$$

The final line above is no greater than α when

$$\frac{\sqrt{\alpha}\log x}{e^{\frac{C_2}{2}E_1}} \leq \frac{1}{\alpha e^{\frac{C_2}{4}E_1}},$$

that is, when x is sufficiently large that

$$\alpha^{3/2} \log x \leq e^{\frac{C_2}{4}E_1}.$$

For such x we have a contradiction to a being nondeficient, hence $u > e^{\frac{C_2}{4\kappa}E_1}$.

Now we factor u into prime powers so that $u = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$. From (A), $p_i^{e_i} < \frac{1}{\sqrt{\alpha}} e^{C_1 E_1}$ if $e_i > 1$, while $p_i^{e_i} < \frac{1}{\sqrt{\alpha}} e^{\frac{C_2}{4\kappa}E_1}$ if $e_i = 1$. Since $C_1 \leq C_2/(4\kappa)$, we have $p_i^{e_i} < \frac{1}{\sqrt{\alpha}} e^{\frac{C_2}{4\kappa}E_1}$ for all i.

Consider the numbers

$$p_1^{e_1}, \quad p_1^{e_1} p_2^{e_2}, \quad \ldots, \quad p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}.$$

Since $u > e^{\frac{C_2}{4\kappa}E_1}$, there is some λ such that

$$p_1^{e_1} p_2^{e_2} \cdots p_\lambda^{e_\lambda} < e^{\frac{C_2}{4\kappa}E_1} \leq p_1^{e_1} p_2^{e_2} \cdots p_{\lambda+1}^{e_{\lambda+1}}.$$

Since $p_{\lambda+1}^{e_{\lambda+1}} < \frac{1}{\sqrt{\alpha}} e^{\frac{C_2}{4\kappa}E_1}$, it follows that

$$p_1^{e_1} p_2^{e_2} \cdots p_{\lambda+1}^{e_{\lambda+1}} < \frac{1}{\sqrt{\alpha}} e^{\frac{C_2}{2\kappa}E_1},$$

so we have found the desired divisor. □

Now we can show that (C) holds. By (A), the squarefull part of the α-pnd is less than $\frac{1}{\sqrt{\alpha}} e^{C_1 E_1}$ and so any divisor satisfying the results of Lemma 5.29 must have squarefree part between $\sqrt{\alpha} e^{\left(\frac{C_2}{4\kappa} - C_1\right)E_1}$ and $\frac{1}{\sqrt{\alpha}} e^{\frac{C_2}{2\kappa}E_1}$.

Next we show that (D) holds. Since a is an α-pnd, $\alpha \leq h(a)$. Let p be the greatest prime factor of the α-pnd a. Comparing (A) and (B), and since $C_1 < C_2$, we see that $p^2 \nmid a$, and so

$$h(a) = h\left(\frac{a}{p}\right)\left(1 + \frac{1}{p}\right).$$

Since a is an α-pnd, $h(a/p) < \alpha$, and so $h(a) < \alpha + \alpha/p$. Then again by (B),

$$h(a) < \alpha + \frac{\alpha}{e^{C_2 E_1}},$$

proving (D).

Finally, we prove that the number r of α-pnd's not greater than x satisfying conditions (A), (B), (C), and (D) is less than

$$\frac{1}{\sqrt{\alpha}} \frac{x}{e^{\left(\frac{C_2}{4\kappa} - C_1\right)E_1}}$$

when x is sufficiently large that it satisfies the bounds of Lemma 5.29. Say they are $a_1, a_2, \ldots, a_r$. From (C), the squarefree part of each a_i has a divisor d_i such that $\sqrt{\alpha} e^{\left(\frac{C_2}{4\kappa} - C_1\right)E_1} < d_i \leq \frac{1}{\sqrt{\alpha}} e^{\frac{C_2}{2\kappa} E_1}$. Therefore,

$$\frac{a_i}{d_i} < \frac{1}{\sqrt{\alpha}} \frac{x}{e^{\left(\frac{C_2}{4\kappa} - C_1\right)E_1}}.$$

We now show that

$$\frac{a_{i_1}}{d_{i_1}} \neq \frac{a_{i_2}}{d_{i_2}}$$

so that the number of integers a_i is the same as the number of integers a_i/d_i so is less than $\frac{1}{\sqrt{\alpha}} x / e^{\left(\frac{C_2}{4\kappa} - C_1\right)E_1}$.

Suppose to the contrary that

$$\frac{a_{i_1}}{d_{i_1}} = \frac{a_{i_2}}{d_{i_2}}.$$

Then $d_{i_1} \neq d_{i_2}$, and

$$h\left(\frac{a_{i_1}}{d_{i_1}}\right) = h\left(\frac{a_{i_2}}{d_{i_2}}\right).$$

Then

$$\frac{h(a_{i_1})}{h(a_{i_2})} = \frac{h(d_{i_1})}{h(d_{i_2})}.$$

Since the d_i are squarefree,

$$h(d_{i_1}) \neq h(d_{i_2}),$$

so we reindex if necessary so that

$$\frac{h(d_{i_1})}{h(d_{i_2})} > 1.$$

Now

$$\frac{h(d_{i_1})}{h(d_{i_2})} = \frac{\sigma(d_{i_1})d_{i_2}}{\sigma(d_{i_2})d_{i_1}}$$

and $h(d_{i_2}) < \alpha$, so

$$\sigma(d_{i_2}) < \alpha d_{i_2},$$

and so the denominator of $\frac{h(d_{i_1})}{h(d_{i_2})}$ is less than $\alpha d_{i_1} d_{i_2}$.

Hence

$$\frac{h(d_{i_1})}{h(d_{i_2})} > 1 + \frac{1}{\alpha d_{i_1} d_{i_2}} > 1 + \frac{1}{\alpha} \cdot \frac{\alpha}{e^{\frac{C_2}{\kappa} E_1}} > 1 + \frac{1}{e^{C_2 E_1}},$$

while from (D), we see that

$$\frac{h(a_{i_1})}{h(a_{i_2})} < \frac{\alpha + \alpha / e^{C_2 E_1}}{\alpha} = 1 + \frac{1}{e^{C_2 E_1}}.$$

This contradicts that the two sides are equal, proving our final bound.

Comparing our two bounds

$$4\sqrt[4]{\alpha} C \frac{x}{e^{\frac{C_1}{2} E_1}} \qquad \text{and} \qquad \frac{1}{\alpha} \frac{x}{e^{\left(\frac{C_2}{4\kappa} - C_1\right) E_1}},$$

we observe that we may choose

$$C_1 = \frac{2}{3} \cdot \frac{C_2}{4\kappa} = \frac{C_2}{6\kappa}.$$

It remains to check that the conditions of Lemma 5.25 are satisfied for some choice of C_2. We find for any value of $\kappa \geq 2$ that any value of C_2 such that

$$0.01388\cdots = \frac{\sqrt{e} - \frac{eC_3}{2}}{C_3 e^{3/2}} \leq C_2 \leq \frac{\sqrt{e} - \frac{1}{2}}{\frac{1}{6} + \sqrt{e}} = 0.6327\ldots$$

will work. Thus, we can define C_2 to be the upper bound, so that

$$\beta := C_2 = \frac{\sqrt{e} - \frac{1}{2}}{\frac{1}{6} + \sqrt{e}} = 0.6327\ldots.$$

Using this value of C_2, we determine C_4:

$$\begin{aligned} C_4 &= \frac{\frac{1}{3}\sqrt{e} + 2e - \frac{1}{3\kappa}\left(\sqrt{e} - \frac{1}{2}\right)}{\frac{1}{6} + 2e} \\ &= 1.06833684\ldots - \frac{0.0683368464\ldots}{\kappa} \\ &\geq 1.03416843. \end{aligned}$$

We now determine a bound on x satisfying each of the bounds in Lemmas 5.26 and 5.29. The bound

$$x \geq \exp\exp\left(\left(\frac{eC_3}{C_4}\right)^2\right)$$

is satisfied when $x \geq \exp(10890)$.

For the bound

$$\frac{\log\log x}{\log\log\log x} \geq \frac{4C_4}{C_4 - C_1},$$

we use the upper bound $C_1 \leq C_2/12$ when $\kappa \geq 2$, and the lower bound $C_4 \geq 1.03416843$. This gives the bound

$$\frac{\log\log x}{\log\log\log x} \geq 4.21491229.$$

By Lemma 5.24, the function

$$\frac{\log\log x}{\log\log\log x}$$

is increases as $x \geq \exp(\exp(e))$ increases, with a minimum value of e. With $x = \exp(13100)$, we have

$$\frac{\log\log x}{\log\log\log x} = 4.214952\ldots.$$

Thus, the inequality is satisfied when $x \geq \exp(13100)$.

To bound

$$\frac{3}{2}\log\alpha + \log\log x \leq \frac{C_2}{4}\sqrt{\log x \log\log x},$$

we use Lemma 5.28 which gives the inequality

$$\log\alpha < \log C_6 + \log\log\log x,$$

along with the bound

$$C_6 < \log\log x$$

when $x > \exp(\exp(C_6)) = \exp(7.90\ldots)$.

Next, we bound

$$3\log\log\log x + \log\log x \leq \frac{C_2}{4}\sqrt{\log x \log\log x},$$

As we have observed, $e \log\log\log x \le \log\log x$ for $x \ge \exp(\exp(e))$, so we need

$$\left(\frac{3}{e}+1\right)\log\log x \le \frac{C_2}{4}\sqrt{\log x \log\log x},$$

which simplifies to

$$\sqrt{\frac{\log x}{\log\log x}} \ge \frac{4}{C_2}\left(\frac{3}{e}+1\right).$$

This bound is satisfied when $x \ge \exp(1300)$.

Finally, we address the bound

$$\sqrt{\log x \log\log x} \ge \frac{6}{C_1}\log\frac{3\sqrt[12]{\alpha}}{C}.$$

Using $C_1 = C_2/(6\kappa)$ and Lemma 5.28, we have that x must be large enough to satisfy

$$\kappa \le \frac{C_2\sqrt{\log x \log\log x}}{36\log\frac{3}{C} + 3\log(C_6\log\log x)}.$$

Thus, we have proven the theorem. □

This theorem allows us to prove the convergence of the sum

$$\sum_{i=1}^{\infty}\frac{\phi(c_i)}{c_i}\cdot\frac{1}{a_i}$$

for any non-Liouville α, using the same proof as for the case $\alpha = 2$. This gives us the following corollary.

Corollary 5.30. *Let $\{a_i\}_{i=1}^{\infty}$ denote the sequence of α-pnd's for α non-Liouville, and* *let* $\{c_i\}_{i=1}^{\infty}$ *be the corresponding cofactor sequence. The infinite sum*

$$\sum_{i=1}^{\infty} \frac{\phi(c_i)}{c_i} \cdot \frac{1}{a_i}$$

converges and the value is equal to $\mathbf{d}\,\mathscr{A}_\alpha$*.*

Remark 5.31. We can see that this sum is indeed infinite for any $\alpha > 1$ by constructing a sequence of α-pnd's. We first note that for sufficiently large primes, say $p_i, i \geq n_0$, where p_i denotes the ith prime, we have

$$h(p_i) = 1 + \frac{1}{p_i} < \alpha.$$

Now there is some r such that

$$h(p_i p_{i+1} \cdots p_{i+r-1}) < \alpha \leq h(p_i p_{i+1} \cdots p_{i+r})$$

by divergence of the sum of prime reciprocals. Then $p_i p_{i+1} \cdots p_{i+r}$ is an α-pnd. In this way we can construct an α-pnd for each $i \geq n_0$.

It remains to consider the case where α is Liouville. We have already shown that the sum of reciprocal α-pnd's may not converge, so that we may not prove convergence of the density sum expression in the same way as above. In [14], Erdős proves that for any α,

$$N_\alpha(x) = o\left(\frac{x}{\log x}\right) \tag{5.13}$$

as $x \to \infty$. In fact, with minor changes to the proof, (5.13) can be shown to hold uniformly in α.

There are two places in the proof which appear to depend on the value of α.

The first occurs in what he calls the first class, second subclass on p. 29. To remove the α dependence, we observe that since $h(n) = O(\log\log x)$, we need only consider $\alpha < c\log\log x$ for some positive constant c. The second place occurs at the top of p. 32. To remove the α dependence here, we need that the reciprocal sum of numbers n such that $h(n) = c'$ for some constant c', is bounded by some C that does not depend on c'. We will use a result of Wirsing [37].

Theorem 5.32. *Let $\mathscr{P}_\alpha$ denote the set of α-perfect numbers. Then*

$$|\mathscr{P}_\alpha(x)| \leq x^{\frac{c}{\log\log x}} \qquad \text{for } x \geq 3$$

for some $c > 0$, where c does not depend on α.

Using this result along with partial summation gives us that the reciprocal sum is bounded by a universal constant C. Thus, we arrive at the following lemma.

Lemma 5.33.

$$N_\alpha(x) = o\left(\frac{x}{\log x}\right)$$

as $x \to \infty$ uniformly in α.

In [11], Erdős also proves the following lemma.

Lemma 5.34. *The number of integers $n \leq x$ that do not satisfy all of the following three conditions:*

(a) if $p^e \mid n$ and $e > 1$, then $p^e < (\log x)^{10}$,

(b) the number of different prime factors of n is less than $10\log\log x$,

(c) the greatest prime factor of n is greater than $x^{1/(20\log\log x)}$,

is $o(x/(\log x)^2)$.

Using these tools, we are now in a position to extend the density sum relation of Corollary 5.30 for α Liouville.

Theorem 5.35. *Let* $\alpha > 1$ *be real, let* a_i *denote the ith* α*-pnd, and let* $c_i = L_k/a_i$*, where* L_k *is defined in* (5.6)*. Then*

$$\mathbf{d}\,\mathscr{A}_\alpha = \sum_{i=1}^{\infty} \frac{\phi(c_i)}{c_i} \cdot \frac{1}{a_i}.$$

Proof. We partition the set of α-pnd's $a_i \le x$ into two classes. In the first class we have those not satisfying all three conditions listed in Lemma 5.34. Since the number of these is $o(x/(\log x)^2)$, the reciprocal sum of these α-pnd's converges.

For the second class consisting of those α-pnd's that do satisfy the conditions listed in Lemma 5.34, we argue as follows. First we note that

$$\frac{\varphi(L_k/a_i)}{L_k/a_i} \cdot \frac{1}{a_i} \le \frac{\varphi(L_k)}{L_k} \cdot \frac{a_i}{\varphi(a_i)} \cdot \frac{1}{a_i} = \frac{\varphi(L_k)}{L_k} \cdot \frac{1}{\varphi(a_i)}.$$

Next we estimate $\varphi(L_k)/L_k$ and $\varphi(a_i)$. By condition (c), we have that a_i, and thus also L_k, contains primes greater than $x^{1/(20\log\log x)}$. Then by definition of L_k and $F(x)$,

$$\frac{\varphi(L_k)}{L_k} \le F(x^{1/(20\log\log x)}) = O\left(\frac{\log\log x}{\log x}\right).$$

By condition (b) we can bound $\varphi(n)/n$ by

$$\frac{\varphi(n)}{n} \ge F(p_{\omega(n)}) \ge F(10\log\log n \log\log\log n) \sim \frac{e^{-\gamma}}{\log\log\log n},$$

where here p_i denotes the ith prime.

Thus, for large x, our a_i satisfy

$$\frac{1}{\varphi(a_i)} \leq \frac{e^{\gamma} \log\log\log x}{a_i}.$$

Putting our estimates together, we find that

$$\frac{\varphi(L_k/a_i)}{L_k/a_i} \cdot \frac{1}{a_i} \leq \frac{f(x)}{a_i}$$

where

$$f(x) = O\left(\frac{\log\log x \log\log\log x}{\log x}\right).$$

Thus, the sum over a_i that are α-pnd's satisfying our conditions is

$$\sum_{a_i \leq x} \frac{\varphi(L_k/a_i)}{L_k/a_i} \cdot \frac{1}{a_i} = O\left(\sum_{a_i \leq x} \frac{\log\log a_i \log\log\log a_i}{a_i \log a_i}\right).$$

Since the number of α-pnd's up to x is bounded by (5.13), the sum converges by partial summation. Thus, we have shown convergence of the sum over α-pnd's in the second class. We conclude that the density expression holds. □

5.8 Organization

We now introduce a method of organizing natural numbers in line with the notion of significance. Consider a natural number $n > 1$ with the canonical factorization

$$n = \prod_{p_i | n} p_i^{e_i}.$$

Taking h of both sides, we have

$$h(n) = \prod_{p_i|n} h(p_i^{e_i}) = \prod_{p_i|n} \prod_{j=1}^{e_i} \frac{h(p_i^j)}{h(p_i^{j-1})}.$$

Now observe that

$$h'(p^j) := \frac{h(p^j)}{h(p^{j-1})} = 1 + \frac{1}{\sigma(p^j) - 1}. \tag{5.14}$$

Thus, $h(n)$ is a product of factors having form $1 + 1/(\sigma(p^e) - 1)$, and the number of such factors is the same as the number of prime factors p counted with multiplicity. If the factors (5.14) are ordered according to decreasing significance of p^j, this ordering induces an ordering on the prime factors of n. We will call this the factorization of n according to prime significance (as opposed to prime power significance). For instance, if $n = 2^3 7^2$, we would factor $h(2^3 7^2)$ as

$$h(2^3 7^2) = h'(2)h'(2^2)h'(7)h'(2^3)h'(7^2),$$

since

$$\text{sig}(2) > \text{sig}(2^2) > \text{sig}(7) > \text{sig}(2^3) > \text{sig}(7^2).$$

Then the induced ordering of prime factors is

$$2^3 7^2 = 2 \cdot 2 \cdot 7 \cdot 2 \cdot 7.$$

Suppose that $n = p_1 \cdots p_k$ is ordered in this way. Write $n_i = p_1 \ldots p_i$, so that in particular $n = n_k$. Then we have that $h(n_i)$ is an α-pnd for any α in the interval $(h(n_{i-1}), h(n_i)]$. This implies that for any choice of α in $[1, h(n)]$, there exists a well-defined divisor of n which is an α-pnd. We will call this the *significant α-pnd of n*, and as usual we may drop the α when $\alpha = 2$.

In the above example of $2^3 7^2$, we have intervals

$$\{h(1)\}, (h(1), h(2)], (h(2), h(2^2)], (h(2^2), h(2^2 7)], (h(2^2 7), h(2^3 7)], (h(2^3 7), h(2^3 7^2)],$$

namely

$$\{1\}, \left(1, \tfrac{3}{2}\right], \left(\tfrac{3}{2}, \tfrac{7}{4}\right], \left(\tfrac{7}{4}, 2\right], \left(2, \tfrac{15}{7}\right], \left(\tfrac{15}{7}, \tfrac{855}{392}\right].$$

Now we can read off the significant α-pnd for $2^3 7^2$ for any α in $[1, h(2^3 7^2)]$. For instance, the significant pnd of $2^3 7^2$ must be $2^2 7$ since 2 falls in the interval $(\frac{7}{4}, 2] = (h(2^2), h(2^2 7)]$. If we instead wanted the significant $\zeta(2)$-pnd, we note that $\zeta(2) = 1.644 \cdots \in (1.5, 1.75] = (h(2), h(2^2)]$, so this would be 2^2.

The notion of significant α-pnd's provides a more natural proof that the density of α-abundants has the form

$$\sum_{a \text{ an } \alpha\text{-pnd}} \frac{\phi(c(a))}{c(a)} \cdot \frac{1}{a}$$

for some function $c(n)$. Namely, we partition the set $\mathscr{A}'_\alpha$ of α-nondeficient numbers according to significant α-pnd of its members. Let $\mathscr{A}'_\alpha[a]$ be the set of α-nondeficient numbers with a as their significant α-pnd. We can factor such a number n as $n = ma$. The only primes which cannot divide m are those primes p such that either $p \nmid a$ and $\operatorname{sig}(p) < \operatorname{sig}(a)$, or $p^e \| a$ and $\operatorname{sig}(p^{e+1}) < \operatorname{sig}(a)$. If we extend the definition of the symbol $\|$ so that $p \nmid n \implies p^0 \| n$, then we can combine the conditions on m so that

p cannot divide m where $p^e \| a$ and $\mathrm{sig}(p^{e+1}) < \mathrm{sig}(a)$. Thus,

$$c(a) = \prod_{\substack{p^e \| a \\ \mathrm{sig}(p^{e+1}) < \mathrm{sig}(a)}} p.$$

Thus, we have that

$$\mathbf{d}\, \mathscr{A}_a^{\alpha'} = \frac{\phi(c(a))}{c(a)} \cdot \frac{1}{a},$$

and the theorem can be completed as before.

5.9 An α-pnd listing algorithm

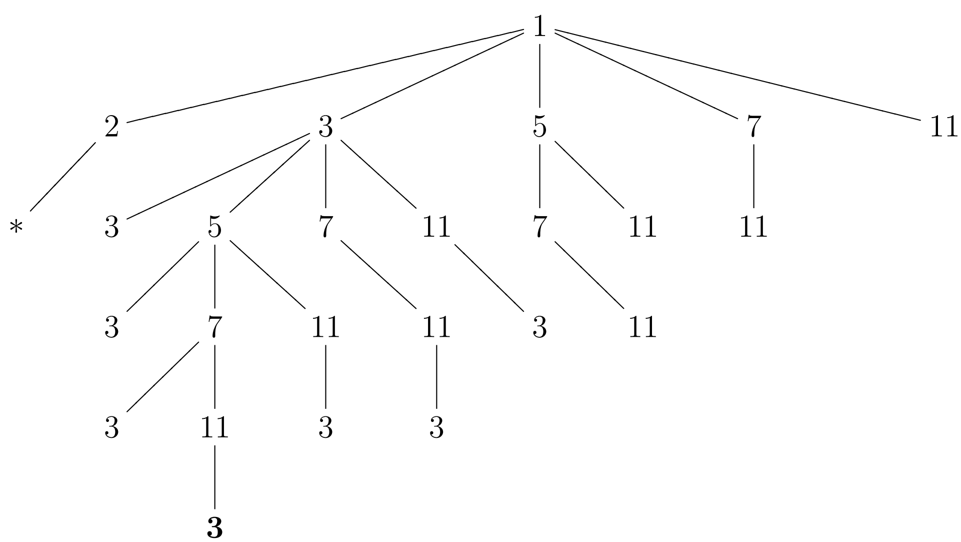
Viewing natural numbers in terms of their prime significance factorization allows us to describe an algorithm for finding all elements of the set $\mathbb{P}_\alpha^k$ of α-pnd's with significance bounded below by $\mathrm{sig}(p_k^{e_k})$, where $p_k^{e_k}$ is the kth term of the sequence P of prime powers ordered by decreasing significance. We recall the notation L_k for the lcm of the first k terms of the sequence P. We now define $L_{j,k}$ to be the lcm of the terms $p_i^{e_i}$ for $i = j+1, \ldots, k$ of P. Thus, $L_{0,k} = L_k$. We will now describe an iterative method that will eventually find all members of $\mathbb{P}_\alpha^k$.

Fix $\alpha \geq 1$. If $\alpha = 1$, then 1 is the only 1-pnd, so $\mathbb{P}_1 = \{1\}$ and we are done. Otherwise $\alpha > 1$. Let $n = 1$ and $i = 1$. The iterative step takes a number n such that $h(n) < \alpha$ and i such that $\mathrm{sig}(n) = \mathrm{sig}(p_i^{e_i})$. Next we determine primes p which may be multiplied to the current value of n so that pn is a potential α-pnd. These are precisely the primes p that do not divide L_i/n but do divide $L_{i,k}$. For each such prime p, if

$$\alpha \in (h(n), h(np)]\,,$$

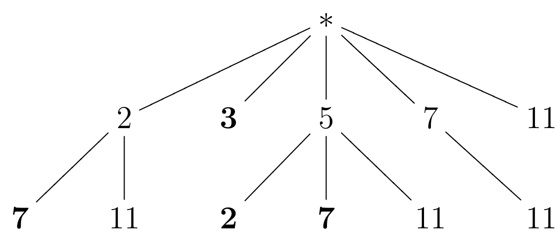
then we include np in our list of α-pnd's. Otherwise $\alpha > h(np)$, and we begin a new iteration with the number np such that $h(np) < \alpha$, and the index i' of the prime power $p_{i'}^{e_{i'}}$ such that $\mathrm{sig}(np) = \mathrm{sig}(p_{i'}^{e_{i'}})$. Once the primes p are exhausted, we return from the iteration. We will call this the *α-pnd listing algorithm*.

To illustrate, we will give the example of the tree found using the α-pnd listing algorithm corresponding to $\alpha = 2$, $p_7^{e_7} = 3^2$. Off of $n = 1$, we build branches for each prime $p \mid L_7 = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$. If there are permissible primes, these primes continue to branch off previous primes. We eventually arrive at the following tree.



The branch terminating with the boldface **3** denotes the discovery of the pnd $3 \cdot 5 \cdot 7 \cdot 11 \cdot 3$, while the other branches have terminated due to using up all allowed primes.

We complete the part of the tree marked by $*$ below.



Again we have some branches terminating in boldface indicating pnd's, and other branches that have run out of usable primes.

However, this tree has so many nodes that it would not be practical to use this in a program, as it will take too long to traverse this tree. We can see this by estimating the number of these as follows. We first determine an upper bound for the number of nodes that must be traversed to find all of the specified α-pnd's. This can be seen by counting the number of nodes on a tree built in the following way: Begin with a root labeled by the number 1. Append branches p_i to the root for $i = 1, \ldots, k$, where k is the number of prime powers having significance bounded by $\mathrm{sig}(p_k^{e_k})$. At each node p_i, append branches p_j, $j = i+1, \ldots, k$. We note that the number of nodes is an upper bound for the number of α-pnd's with significance bounded by $\mathrm{sig}(p_k^{e_k})$. Observe that the number of nodes in this tree is the same as the number of subsets that can be formed from k objects, so there are 2^k nodes. We bound k by $\pi(2y)$, where we set $y = p_k^{e_k}$. This can be seen by noting first that k is less than the number of prime powers bounded by $2y$. We have $p^e \leq 2y$ so $e \leq \log 2y / \log p$. Thus, we must bound

$$\log 2y \sum_{p \leq 2y} \frac{1}{\log p}.$$

By partial summation this is $O(y/\log y)$. Thus, the algorithm will eventually find all of the specified α-pnd's by traversing at most $2^{O(y/\log y)}$ nodes.

5.10 The α-pnd density algorithm

Rather than using the tree in the previous section to identify the α-pnd's, we have the following alternate method. We will want to compute the prime factorization of each α-pnd a_i up to some bound z. To do this, we can use a modified sieve of Eratosthenes to identify the prime factorizations of numbers up to z. This can be done in $O(z \log\log z)$ steps. Simultaneously, we can keep track of the h value of each number as well as the σ value of the prime power factors. We check for α-abundancy and discard the α-deficient numbers. Of the remaining numbers, we identify the least significant prime power p^e and calculate $h(a/p)$ to determine primitivity. This amounts to checking for the largest σ value of prime power factors of n.

We also wish to compute the c_i corresponding to a_i. In fact, c_i is very large in general so we find the value $\varphi(c_i)/c_i$ instead. In preparation, we set up an array of values of $\varphi(L_k)/L_k$, where the largest k needed is determined by determining the largest k satisfying

$$\sigma(p_k^{e_k}) = \max\{\sigma(p^e) : p^e \le z\}.$$

This array has $O(\pi(z))$ entries. To find $\varphi(c_i)/c_i$, we begin with $\varphi(L_k)/L_k$, for the k corresponding to a_i. Then we adjust this value with the prime powers in a_i that divide L_k to the highest power that L_k has. The appropriate prime powers $p^e \| a_i$ can be found by checking that they satisfy $\mathrm{sig}(p^{e+1}) < \mathrm{sig}(a_i)$, where $\mathrm{sig}(a_i)$ is the significance of the least significant prime power dividing a_i. Once $\varphi(c_i)/c_i$ is determiend, it is multiplied to $1/a_i$ and is added on to a running sum to determine

$$\sum_{a_i \le z} \frac{\varphi(c_i)}{c_i a_i}. \tag{5.15}$$

The multiplications involved take $O((\log z)^2)$ steps, as discussed in Subsection 3.3.1. Thus the time spent calculating a lower bound for the density of α-abundant numbers is

$$O\left(z(\log z)^2 \log\log z\right). \tag{5.16}$$

This bound, along with the calculation for Theorem 5.15, allows us to determine an upper bound for the running time of the α-pnd algorithm. We will now repeat this argument more carefully by making everything explicit. In what follows, we will refer to the upper bound estimate of the tail sum

$$\sum_{a_i > z} \frac{\varphi(c_i)}{c_i a_i}$$

as the *error of the α-pnd algorithm*, and the parameter z as the *α-pnd bound*. The value of the error is what we must add to the truncated sum lower bound (5.15) to arrive at an upper bound for the density of the α-abundants.

Theorem 5.36. *For α non-Liouville, the error of the α-pnd algorithm with α-pnd bound z is bounded by*

$$\sum_{a_i > z} \frac{\varphi(c_i)}{c_i a_i} \le -\frac{|\mathbb{P}_\alpha(z)|}{z} + \frac{12\delta\kappa}{\beta - \frac{6\kappa}{(\log z \log\log z)^{1/2}}} \left(\frac{\log z}{\log\log z}\right)^{1/2} \exp\left(-\frac{\beta}{12\kappa}(\log z \log\log z)^{1/2}\right)$$

where the sum is over α-pnd's $a_i > z$, $c_i = L_k/a_i$, where L_k is defined in (5.6), $\mathbb{P}_\alpha$ *is the set of α-pnd's, and β, δ, and κ are defined in Theorem 5.20.*

Proof. We use partial summation and Theorem 5.20.

$$\begin{aligned}\sum_{a>z} \frac{1}{a} &= -\frac{|\mathbb{P}_\alpha(z)|}{z} + \int_z^\infty \frac{|\mathbb{P}_\alpha(t)|}{t^2} dt \\ &\leq -\frac{|\mathbb{P}_\alpha(z)|}{z} + \delta \int_z^\infty \frac{1}{te^{\beta/(12\kappa)(\log t \log\log t)^{1/2}}}.\end{aligned}$$

To bound the integral in the last line above, we use the integral

$$\begin{aligned}c\int_z^\infty \left(1 + \frac{1}{\log\log t}\right) \frac{dt}{te^{c(\log t \log\log t)^{1/2}}} &= \left(\frac{\log z}{\log\log z}\right)^{1/2} \frac{1}{e^{c(\log z \log\log z)^{1/2}}} \\ &+ \frac{1}{2}\int_z^\infty \frac{1}{t(\log t \log\log t)^{1/2}} \left(1 - \frac{1}{\log\log t}\right) \frac{dt}{e^{c(\log t \log\log t)^{1/2}}}.\end{aligned}$$

From this, we have the bound

$$\begin{aligned}\int_z^\infty \frac{dt}{te^{c(\log t \log\log t)^{1/2}}} &\leq \frac{1}{c}\left(\frac{\log z}{\log\log z}\right)^{1/2} \frac{1}{e^{c(\log z \log\log z)^{1/2}}} \\ &+ \frac{1}{2c\left(\log z \log\log z\right)} \int_z^\infty \frac{dt}{te^{c(\log t \log\log t)^{1/2}}}.\end{aligned}$$

By solving this inequality for our desired integral, we have

$$\int_z^\infty \frac{dt}{te^{c(\log t \log\log t)^{1/2}}} \leq \frac{1}{c\left(1 - \frac{1}{2c(\log z \log\log z)^{1/2}}\right)} \left(\frac{\log z}{\log\log z}\right)^{1/2} \frac{1}{e^{c(\log z \log\log z)^{1/2}}}.$$

This gives us our result. □

Now suppose we want the error to be within 10^{-d} for some d. For α non-Liouville, we have by Theorem 5.36 that it suffices to have

$$10^{-d} \leq \frac{12\delta\kappa}{\beta - \frac{6\kappa}{(\log z \log\log z)^{1/2}}} \left(\frac{\log z}{\log\log z}\right)^{1/2} \frac{1}{e^{\frac{\beta}{12\kappa}(\log z \log\log z)^{1/2}}},$$

with bounds on z and κ as stated in Theorem 5.36. In particular, we have the bounds

$$\kappa \leq \beta \frac{(\log z \log\log z)^{1/2}}{36 \log \frac{3\zeta(3)}{\zeta(3/2)} + 3\log(\eta \log\log z)}$$

and $z \geq \exp(13100)$. We use these to bound

$$\frac{12\delta\kappa}{\beta - \frac{6\kappa}{(\log z \log\log z)^{1/2}}} \leq \frac{1}{1 - \frac{1}{6\log\frac{3\zeta(3)}{\zeta(3/2)} + \frac{1}{2}\log(\eta\log(13100))}} \frac{12\delta\kappa}{\beta}.$$

We will let

$$\lambda = \frac{1}{1 - \frac{1}{6\log\frac{3\zeta(3)}{\zeta(3/2)} + \frac{1}{2}\log(\eta\log(13100))}} = 1.4128\ldots.$$

It thus suffices to have

$$10^{-d} \leq \frac{12\delta\lambda\kappa}{\beta} \left(\frac{\log z}{\log\log z}\right)^{1/2} \frac{1}{e^{\frac{\beta}{12\kappa}(\log z \log\log z)^{1/2}}}.$$

Taking logs and multiplying by -1, we get

$$(\log 10)d \geq -\log\left(\frac{12\delta\lambda\kappa}{\beta}\right) - \frac{1}{2}\log\left(\frac{\log z}{\log\log z}\right) + \frac{\beta}{12\kappa}(\log z \log\log z)^{1/2}.$$

Noting that $12\delta\lambda\kappa/\beta > 1$ for all $\alpha \geq 1$, it suffices to have

$$(\log 10)d \geq \frac{\beta}{12\kappa}(\log z \log\log z)^{1/2},$$

or

$$z^{\log\log z} \le e^{\left(\frac{12(\log 10)\kappa d}{\beta}\right)^2}.$$

Since the running time t is

$$O\left(z(\log z)^2 \log\log z\right),$$

for some constant c we have

$$t < cz(\log z)^2 \log\log z.$$

We now use the bounds

$$\log z \log\log z \le \left(\frac{12(\log 10)\kappa d}{\beta}\right)^2$$

and

$$z\log z = e^{\log z + \log\log z} < e^{\log z \log\log z} \le e^{\left(\frac{12(\log 10)\kappa d}{\beta}\right)^2}$$

to arrive at the following result.

Theorem 5.37. *For α non-Liouville, the α-pnd algorithm described above can determine the density of abundant numbers to d decimal digits in at most t time, where*

$$t < c\left(\frac{12(\log 10)\kappa d}{\beta}\right)^2 e^{\left(\frac{12(\log 10)\kappa d}{\beta}\right)^2},$$

where c is an absolute constant and β and κ are defined in Theorem 5.20.

For α Liouville, we use Lemma 5.33. Again, we use partial summation to find the following.

Lemma 5.38. *For α Liouville, the error bound for the α-pnd method with α-pnd bound z is*

$$\sum_{a_i>z} \frac{\varphi(c_i)}{c_i a_i} = O\left(\frac{\log\log z \log\log\log z}{\log z}\right),$$

where the sum is over α-pnd's $a_i > z$, $c_i = L_k/a_i$, and L_k is defined in (5.6).

By this lemma, we see that for the error to be within 10^{-d}, we must have

$$d > \log\log z.$$

Then with the same time bound

$$t < cz(\log z)^2 \log\log z,$$

we find the following.

Theorem 5.39. *For α Liouville, the α-pnd algorithm can determine the density of abundant numbers to d decimal digits in at most t time, where*

$$t < cde^{2d}e^{e^d},$$

where c is an absolute constant.

Thus we find that when α is Liouville, the time grows at worst double exponentially with the number of desired digits, just as we found for the Deléglise algorithm. In contrast, when α is non-Liouville, we have an improved bound of time growing at worst single exponentially with the number of desired digits.

5.11 A result of Shapiro

In [32], Harold Shapiro proves that if there are infinitely many α-p.n.d's with k distinct prime factors, then α can be expressed as

$$\alpha = \frac{\sigma(a)}{a} \cdot \frac{b}{\phi(b)}, \qquad (a, b) = 1, \qquad b > 1, \tag{5.17}$$

and $\omega(a)+\omega(b) < k$. That these conditions are also sufficient was proven by Shapiro in [33]. Using our theory of significance, we provide a streamlined proof of the sufficiency of Shapiro's theorem. In preparation, we build two tools which may be of independent interest.

5.11.1 The capping off lemma

We first characterize when an α-deficient number a' can be augmented by a prime power p^e with $\mathrm{sig}(p^e) < \mathrm{sig}(a')$ so that $a'p^e$ is an α-pnd.

Lemma 5.40 (The capping-off lemma). *Given a number $a' = p_1^{e_1} \cdots p_k^{e_k}$ with $p_k^{e_k}$ having minimal significance in a', and a real $\alpha > h(a')$, $a'p$ is an α-pnd with* $\mathrm{sig}(p) <$ $\mathrm{sig}(a')$ *if and only if p is a prime in the interval*

$$\sigma(p_k^{e_k}) - 1 < p \leq \frac{h(a')}{\alpha - h(a')}.$$

In addition, for any α-deficient a' there is at most one prime power $p^e, e > 1$, such that $(p, a') = 1$, and $a = a'p^e$ is an α-pnd. In fact, the prime must lie in the interval

$$\frac{h(a')}{\alpha - h(a')} < p < \frac{h(a')}{\alpha - h(a')} + 1.$$

If there is such a prime p, then the exponent e is

$$e = \left\lceil \frac{\log\left(\frac{h(a')}{h(a')-\alpha(1-1/p)}\right)}{\log p} - 1 \right\rceil,$$

where $\lceil x \rceil$ denotes the ceiling of x.

For any α-pnd a there is a number a' such that for some prime power p^e with $\mathrm{sig}(p^e) < \mathrm{sig}(a')$, $a = a'p^e$.

Proof. Note that a number $a'p$ is an α-pnd with $\mathrm{sig}(a') > \mathrm{sig}(p)$ if and only if $h(a'p) \geq \alpha$ and $\sigma(p_k^{e_k}) < \sigma(p)$. We show that the latter inequality is strict since it is not possible for a prime p to have the same σ value as a prime power distinct from p. For suppose not and $\sigma(p) = \sigma(q^e)$ for a prime power q^e. If $q = p$, then $e > 1$, but this means $\sigma(q^e)$ is strictly greater than $\sigma(p)$. Otherwise $q \neq p$. But then

$$p + 1 = q^e + q^{e-1} + \cdots + q + 1,$$

which is absurd since, upon subtracting 1, the right side is divisible by q while the left is equal to p. Thus, we have $\sigma(p_k^{e_k}) < \sigma(p)$. We now solve each of the inequalities

$$h(a')\left(1 + \frac{1}{p}\right) \geq \alpha \qquad \text{and} \qquad \sigma(p_k^{e_k}) < p + 1$$

for p, yielding

$$\sigma(p_k^{e_k}) - 1 < p \leq \frac{h(a')}{\alpha - h(a')}.$$

Suppose $a = a'p^e$ is an α-pnd with $e > 1$ and $(p, a') = 1$. Since a is an α-pnd, we have

$$h(a'p^{e-1}) < \alpha \leq h(a'p^e).$$

Then

$$h(p^{e-1}) < \frac{\alpha}{h(a')} \leq h(p^e),$$

and it is clear that at most one power e for a given prime p can satisfy the above inequalities since for $e = 2, 3, \ldots,$ the intervals $(h(p^{e-1}), h(p^e)]$ partition the interval $(h(p), p/(p-1))$. To see that two primes cannot have overlapping intervals, we write the interval for p as $(1 + 1/p, 1 + 1/(p-1))$. Thus, there is at most one prime power p^e that satisfies our conditions. Solving the inequalities

$$1 + \frac{1}{p} < \frac{\alpha}{h(a')} < 1 + \frac{1}{p-1},$$

for p, we arrive at

$$\frac{h(a')}{\alpha - h(a')} < p < \frac{h(a')}{\alpha - h(a')} + 1.$$

Solving the inequalities

$$\frac{1 - \frac{1}{p^e}}{1 - \frac{1}{p}} < \frac{\alpha}{h(a')} \leq \frac{1 - \frac{1}{p^{e+1}}}{1 - \frac{1}{p}}$$

for e gives

$$\frac{\log\left(\frac{h(a')}{h(a') - \alpha(1-1/p)}\right)}{\log p} - 1 \leq e < \frac{\log\left(\frac{h(a')}{h(a') - \alpha(1-1/p)}\right)}{\log p}.$$

The final statement can be seen by writing $a = a'p_k^{e_k}$, where $\mathrm{sig}(p_k^{e_k}) = \mathrm{sig}(a)$. Since we know that $p_k^{e_k}$ exists, it can be found by one of the two methods of capping-off, depending on whether $e_k = 1$ or not. □

Remark 5.41. Note that not all α-deficient numbers a' have a prime p such that

$\text{sig}(a') > \text{sig}(p)$ and $a'p$ is an α-pnd. An interesting example is when $\alpha = 2$ and we choose $a' = 2^n$. Then we must find a prime p such that

$$2^{n+1} - 2 < p \leq 2^{n+1} - 1,$$

namely a prime $p = 2^{n+1} - 1$. A number $M_n := 2^n - 1$ is called a *Mersenne number*, and if M_n is prime, it is called a *Mersenne prime*. It is known that M_n is not prime unless n is prime, so we cannot always find a prime p to "cap off" $a' = 2^n$. In the event that there is a prime p to cap off 2^n, the pnd $2^n p$ is in fact a perfect number, as proven by Euclid. The pair $\alpha = 2$, $a' = 2^n$ also gives us an example of an a' which cannot be capped off by p^e, $e > 1$, since there is no prime p in the interval

$$2^{n+1} - 1 < p < 2^{n+1}$$

(as there is not even an integer in it).

Now suppose we choose $\alpha = 2$ and $a' = q$, where q is prime. Then we seek primes p such that

$$q < p \leq 1 + \frac{2}{q-1}.$$

This inequality is satisfied only when $q = 2$, in which case we have the prime $p = 3$. We have found the pnd 6, which is also perfect. We now try to cap off a prime by a prime power p^e, $e > 1$. We must find when there is a prime p with

$$1 + \frac{2}{q-1} < p < 2 + \frac{2}{q-1}.$$

For $q = 2$ and 3, the interval does not contain integers. For $q \geq 5$, the interval contains only the prime $p = 2$. Using the inequality

$$\frac{1-\frac{1}{p^e}}{1-\frac{1}{p}} < \frac{\alpha}{h(a')} \leq \frac{1-\frac{1}{p^{e+1}}}{1-\frac{1}{p}},$$

we find that

$$2^e < q+1 < 2^{e+1}.$$

Thus, for instance, if $q = 5$, then $e = 2$ so 20 is a pnd (but not a perfect one).

We have just found all pnd's a with $\omega(a) = 2$. To see this, first we note that $\lim_{e\to\infty} h(3^e)h(5^e) = 3/2 \cdot 5/4 < 2$, so a cannot be odd. So we write $a = 2^e p$ for p a prime greater than 2 and check when

$$h(2^{e-1}p) < 2 \leq h(2^e p)$$

is satisfied. Solving for $p+1$, we find

$$2^e < p+1 \leq 2^{e+1},$$

which is covered by the two classes of pnd's we have found.

5.11.2 The offspring lemma

We now describe a method to iteratively find an infinite sequence of α-pnd's, each member being used to find the next α-pnd, with the number of distinct prime factors increasing by 1 at each step.

Lemma 5.42 (The offspring lemma)**.** *For each α-pnd a_1, there exists an α-pnd a_2 such that $\omega(a_2) = \omega(a_1) + 1$. We can construct a_2 depending on the exponent e of* *the least significant prime power* p^e *of* a_1 *and the exponent* e_1 *of the prime power* $P(a_1)^{e_1} \| a_1$. *If* $e = 1$, *either (1) or (2) may be used. If* $e > 1$ *and* $e_1 = 1$, *we use (2a). Otherwise* $e > 1$ *and* $e_1 > 1$, *in which case we use (2b).*

(1) *If* $e = 1$, *then we can choose* $a_2 = (a_1/p)q_1q_2$ *where* q_1 *is the smallest prime such that* $(a_1/p)q_1$ *is* α*-deficient, and* q_2 *is the prime after* q_1.

(2) *For any* $e \geq 1$, *we split into two subcases, depending on whether the exponent* e_1 *of the prime* $P(a_1)$ *in* a_1 *is 1 or not.*

   (a) *If* $e_1 = 1$, *then we can choose* a_2 *to be the canonical* α*-pnd dividing* $(a_1/P(a_1))q_1q_2$, *where* q_1 *is the smallest prime such that* $(a_1/P(a_1))q_1$ *is* α*-deficient, and* q_2 *is the prime after* q_1.

   (b) *If* $e_1 > 1$, *then we can choose* a_2 *to be the canonical* α*-pnd dividing* $(a_1/P(a_1))q_1$, *where* q_1 *is the largest prime below* $\sigma(P(a_1)^{e_1})$.

Proof. We begin with the first case when $e = 1$. Write $a_1' = a_1/p$. Since $a_1'q_1$ is α-deficient, $q_1 > p$, so $\text{sig}(q_2) < \text{sig}(a_1'q_1)$. Thus, it remains to show that $a_1'q_1q_2$ is α-abundant. Suppose not. Then

$$h(a_1'q_1q_2) < \alpha \leq h(a_1'p),$$

so in particular

$$\left(1 + \frac{1}{q_1}\right)\left(1 + \frac{1}{q_2}\right) = h(q_1q_2) < h(p) = 1 + \frac{1}{p}.$$

However, we have the following result which contradicts this inequality, thus establishing the lemma for the case $e = 1$.

Lemma 5.43. *Let $p_0 < p_1 < p_2$ be consecutive primes. Then*

$$1 + \frac{1}{p_0} < \frac{p_0}{p_0 - 1} < \left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{p_2}\right).$$

In addition, for $p_0 = 2, 3, 5$,

$$1 + \frac{1}{p_0} < \left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{p_2}\right).$$

Proof. The first inequality is evident by cross-multiplication. To prove the second inequality we rely on the following two-prime variant of Bertrand's postulate due to Ramanujan [28], namely

$$\pi(x) - \pi(x/2) \geq 2, \qquad x \geq 11.$$

Then letting $x = 2p_0$ we have $p_2 \leq 2p_0$. Moreover, since p_0 and p_2 are prime, $p_2 \leq 2p_0 - 1$, or $p_0 - 1 \geq (p_2 - 1)/2$. Thus,

$$1 + \frac{1}{p_0 - 1} < 1 + \frac{2}{p_2 - 1} < \left(1 + \frac{1}{p_2 - 2}\right)\left(1 + \frac{1}{p_2}\right) \leq \left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{p_2}\right),$$

establishing the lemma for $p_0 \geq 11/2$. The final inequality may be verified by direct calculation. □

We now turn to the second case where $e \geq 1$. Let $a_1' = a_1/P(a_1)$. For the first subcase where $e_1 = 1$, we can repeat the argument proving case 1 to show that $a_1'q_1q_2$ is α-abundant. Thus, it remains to show that the canonical α-pnd dividing $a_1'q_1q_2$ contains the prime q_2. But any divisor of $a_1'q_1q_2$ not containing q_2 is deficient, so we are done with this subcase.

For the second subcase where $e_1 > 1$, we first show that $a_1'q_1$ is α-abundant. Since $h(a_1')h(q_1) > h(a_1)$, it remains to show that $q_1 \nmid a_1'$. But by Bertrand's postulate,

$$q_1 > \frac{\sigma(P(a_1)^{e_1})}{2} \geq \frac{\sigma(P(a_1)^2)}{2} > P(a_1).$$

We now argue as in the first subcase that the canonical α-pnd dividing $a_1'q_1$ must contain q_1, since any divisor of a_1' is deficient. This completes the proof of the lemma.

□

The offspring lemma allows us, for instance, to find an infinite sequence of squarefree pnd's by starting with a squarefree pnd and using construction (1). Thus,

$$2 \cdot 3, \quad 2 \cdot 5 \cdot 7, \quad 2 \cdot 5 \cdot 11 \cdot 13, \quad 2 \cdot 5 \cdot 11 \cdot 59 \cdot 61, \ldots$$

are all pnd's.

5.11.3 A proof of Shapiro's theorem

We are now prepared to prove Shapiro's theorem. Suppose there are infinitely many α-pnd's composed of k primes. We arrange them in order of significance. Then, taking a_i to be the ith α-pnd in this sequence, and $\mathrm{sig}(a_i) = \mathrm{sig}(p_k^{e_k})$, we have

$$h(a_i)\left(1 - \frac{1}{\sigma(p_k^{e_k})}\right) = h(a_i)\left(\frac{h(p_k^{e_k-1})}{h(p_k^{e_k})}\right) < \alpha \leq h(a_i).$$

Since $\sigma(p_k^{e_k}) \to \infty$, we have $h(a_i) \to \alpha$. We now determine $\lim_{i\to\infty} h(a_i)$. We factor a_i as

$$a_i = p_{i1}^{e_{i1}} \cdots p_{ik}^{e_{ik}},$$

where the primes p_{ij} are decreasing in j. Now for each j we examine the sequence in i of primes p_{ij}. Let k' be the smallest j such that there is a constant subsequence. Thus, $\lim_{i\to\infty} h(p_{ij}^{e_{ij}}) = 1$ for $j < k'$. Now we pass to this subsequence, and call the prime constant $p_{k'}$, so that now $p_{ik'} = p_{k'}$. Since the primes of a_i are in decreasing order, $p_{i(k'-1)} < p_{k'}$, so we may now pass to an infinite subsequence where $p_{i(k'-1)}$ is constant. This process is continued until we have a subsequence where p_{ij} is constant in i for all $j \geq k'$.

Now we examine the e_{ij} for $i \geq k'$. For each e_{ij} that is unbounded in i, we pass to a subsequence where $\lim_{i\to\infty} e_{ij} = \infty$. Then we have $h(p_j^{e_{ij}}) \to p_j/(p_j - 1) = p_j/\varphi(p_j)$. The product of these primes is b.

The remaining primes have a sequence e_{ij} bounded in i. Thus, we can pass to an infinite subsequence such that e_{ij} is constant in i. We call this constant e_j. Then the product of these prime powers $p_j^{e_j}$ is a, and we have $\alpha = h(a) \cdot b/\varphi(b)$, as claimed.

We now prove sufficiency. Let α satisfy Equation (5.17). Since the function $n/\phi(n)$ is multiplicative and for each prime p, $p^e/\phi(p^e)$ is constant over all $e \geq 1$, we can assume that the given b is squarefree. Noting that as $e \to \infty$, we have $h(b^e) \nearrow \frac{b}{\phi(b)}$, we also have as $e \to \infty$ that $h(ab^e) \nearrow \alpha$, so defining x by

$$1 + \frac{1}{x} = \frac{\alpha}{h(ab^e)},$$

any prime $p \leq x$ and $(p, ab) = 1$ will make $ab^e p$ α-abundant. Let e be sufficiently large that $x > 2\sigma(a)P(b)$. Then by Bertrand's postulate, we can choose p to be any prime in the interval $(\sigma(a)P(b), x]$. Let a_1 be the significant α-pnd of $ab^e p$. We have that all primes dividing $ab^e p$ must divide a_1 since if not, $p \nmid a_1$, contradicting that ab^e is α-deficient. We also have that $a \mid a_1$ since $\mathrm{sig}(a) > \mathrm{sig}(p)$. By the offspring lemma, since we have found an α-pnd with k prime factors, there are α-pnd's for any number of prime factors greater than k. This proves anew the sufficiency part of Shapiro's theorem.

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