MATH 22: HW 3 - Solutions

[Due Oct 6]

Written homework is intended to help students develop their communication and exposition skills through complete write-ups. While correctness of the solution is, of course, necessary, much of the grade for the problem is dependent on clear and appropriate exposition. Exposition shall be appropriate for the type and level of the problem. One principle we use is that exposition should be detailed around the main aspects of the problem, but terse exposition is appropriate for subsidiary parts of a problem.

1. problem 2.4.1

Solution:
BA is 5 by 5

AB is 3 by 3

ABD is 3 by 1

DBA is not allowed

$A(B+C)$ is not allowed

2. problem 2.4.6

Solution:
We have

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}, A+B = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}$$
$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, B^2 = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}, (A+B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}$$

and

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$$

which is not equal to $(A+B)^2$. The correct formula is

$$(A+B)^2 = A^2 + AB + BA + B^2$$

3. problem 2.4.17

Solution:
(a)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \end{bmatrix}$$

4. problem 2.5.1

Solution:

$$A^{-1} = \frac{1}{-12}\begin{bmatrix} 0 & -3 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$$

5. problem 2.5.2

Solution:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

6. problem 2.5.6

Solution:

(a)

$$\begin{aligned} AB = AC &\Rightarrow A^{-1}AB = A^{-1}AC \\ &\Rightarrow B = C \end{aligned}$$

(b) Let $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, C = \begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 0 & 5 \\ 0 & 5 \end{bmatrix}, \quad AC = \begin{bmatrix} 0 & 5 \\ 0 & 5 \end{bmatrix}$$

7. problem 2.5.23

Solution:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & \frac{-1}{2} & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & \frac{-1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{-2}{3} & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{-3}{4} & \frac{3}{2} & -\frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{-2}{3} & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{-3}{4} & \frac{3}{2} & -\frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{-2}{3} & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{-1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{-1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{-1}{2} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

8. problem 2.6.5

Solution: Elimination:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

9. problem 2.6.6

Solution:

$$E_{21}\begin{bmatrix}1&1&1\\2&4&5\\0&4&0\end{bmatrix}=\begin{bmatrix}1&1&1\\0&2&3\\0&4&0\end{bmatrix}$$

$$E_{21}=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}$$

$$E_{32}\begin{bmatrix}1&1&1\\0&2&3\\0&4&0\end{bmatrix}=\begin{bmatrix}1&1&1\\0&2&3\\0&0&-6\end{bmatrix}$$

$$L=E_{21}^{-1}E_{32}^{-1}=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-4&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\2&1&0\\0&-4&1\end{bmatrix}$$

$$A=LU=\begin{bmatrix}1&0&0\\2&1&0\\0&-4&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&2&3\\0&0&-6\end{bmatrix}$$

10. problem 2.6.8

Solution:
(a)

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}$$

$$E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & -c & 1 \end{bmatrix}$$

$$EA = I$$

(b)

$$\begin{aligned} E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} \\ &= \begin{bmatrix} a & 0 & 0 \\ b & c & 1 \end{bmatrix} \end{aligned}$$

11. problem 2.7.1

Solution:

For $A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}$ we have

$$A^\top = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -9 & 1 \end{bmatrix}$$

$$\left(A^{-1}\right)^\top = \frac{1}{3}\begin{bmatrix} 3 & -9 \\ 0 & 1 \end{bmatrix}$$

$$\left(A^\top\right)^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -9 \\ 0 & 1 \end{bmatrix}$$

For the matrix $A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$

$$A^\top = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{c^2}\begin{bmatrix} 0 & -c \\ -c & 1 \end{bmatrix}$$

$$\left(A^{-1}\right)^\top = \frac{-1}{c^2}\begin{bmatrix} 0 & -c \\ -c & 1 \end{bmatrix}$$

$$\left(A^\top\right)^{-1} = \frac{-1}{c^2}\begin{bmatrix} 0 & -c \\ -c & 1 \end{bmatrix}$$

12. problem 2.7.31

Solution:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Total cost: $x^T A^\top y$ Values of one truck and one plane:

$$\begin{bmatrix} 1 & 4 & 2 \\ 50 & 1000 & 50 \end{bmatrix} \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix}$$

$$= \begin{bmatrix} 700 + 12 + 6000 \\ 35000 + 3000 + 15000 \end{bmatrix}$$

1 truck: 6712

1 plane: 53000