MATH 22: HW5 - Solutions

[Due Oct 20]

Written homework is intended to help students develop their communication and exposition skills through complete write-ups. While correctness of the solution is, of course, necessary, much of the grade for the problem is dependent on clear and appropriate exposition. Exposition shall be appropriate for the type and level of the problem. One principle we use is that exposition should be detailed around the main aspects of the problem, but terse exposition is appropriate for subsidiary parts of a problem.

1. problem 3.2.1

Solution:

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$R_2 : R_1 - R_1 \rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$R_3 : R_3 - R_2 \rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1 and x_3 are pivot variables and x_2, x_4 and x_5 are free variables.

$$B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$$

$$R_3 : R_3 - 2R_2$$

$$\rightarrow \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

x_1 and x_2 are pivot variables and x_3 is the free variable.

2. problem 3.2.2

Solution:

(i) Special Solutions for A

$$\begin{cases} x_1 + 2x_2 + 2x_3 + 4x_4 + 6x_5 = 0 \\ \qquad\qquad\quad x_3 + 2x_4 + 3x_5 = 0 \end{cases}$$

Set $x_2 = 1, x_4 = 0, x_5 = 0 \longrightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ Set $x_2 = 0, x_4 = 1, x_5 = 0 \rightarrow \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ Set $x_2 = 0, x_4 = 0, x_5 = 1 \rightarrow \begin{bmatrix} -6 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

(ii) Special solutions for B

$$\begin{cases} 2x_1 + 4x_2 + 2x_3 = 0 \\ \qquad\quad 4x_2 + 4x_3 = 0 \end{cases}$$

set $x_3 = 1$, we get the solution $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

3. problem 3.2.3

Solution:
All Solutions of $Ax = 0$ are given by

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -6 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

where x_2, x_4 and x_5 can be any real number. All solutions of $Bx = 0$,

$$x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad x_3 \in \mathbb{R}.$$

The null space is zero only when there is no free variables.

4. problem 3.2.16

Solution:
Since the null space is zero after reduction there is no free column hence we have 5 pivot variables. If we have 5 pivot columns then the system $Ax = b$ have solution for any b and the column space of A is the entire $\mathbb{R}^5$.

5. problem 3.3.1

Solution:
(a) correct

(b) Incorrect

(c) Correct

(d) Incorrect

6. problem 3.3.2

Solution:
(a)

$$A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $A = 1$ (b)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $A = 2$ (c)

$$A = \begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $A = 1$

7. problem 3.4.1

Solution:

(1)

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & -1 & -1 & -2 & b_3 - b_1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{array}\right] = \left[\begin{array}{cc} u & c \end{array}\right]$$

(2) For solvability we need $b_2 + b_3 - 2b_1 = 0$

(3) The column contains all vectors b such that $Ax = b$ has solution. Plane: $y + z - 2x = 0$

(4) Null space of A

$$\begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Pivot variables: x_1, x_2 Free variables: x_3, x_4 Solution to $Ax = 0$

$$\begin{cases} 2x_1 = -4x_2 - 6x_3 - 4x_4 \\ x_2 = -x_3 - 2x_4 \end{cases}$$

$$\to \begin{cases} x_1 = -x_3 + 2x_4 \\ x_2 = -x_3 - 2x_4 \end{cases}$$

$$\text{Null space } = \left\{ x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} ; x_3, x_4 \in \mathbb{R} \right\}$$

(5) To find particular solution x_p we use back substitution in $Ux = c$

$$\begin{cases} 2x_1 + 4x_2 + 6x_3 + 4x_4 = 4 \\ \qquad x_2 + x_3 + 2x_4 = -1 \\ \qquad\qquad 0 = 3 + 5 - 2(4) = 0 \end{cases}$$

set free variables to zero and we get

$$x_p = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Solution: [continue]
The complex solution is given by adding a vector in null space to x_p :

$$\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad x_3, x_4 \in \mathbb{R}$$

(6) Reduced form

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} R & d \end{bmatrix}$$

special solution from R

$$x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

particular solution from d :

$$x_p = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

8. problem 3.4.4

Solution:
Reduction steps:

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The null space is given by

$$x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

and the particular solution is

$$x_p = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

hence all solutions are of the form

$$x = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

9. problem 3.5.1

Solution:
We define a matrix with v_1, v_2, v_3, v_4 as columns.

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Given $Ax = 0$, the vectors v_1, v_2, v_3, v_4 are independent if zero is the only solution to $Ax = 0$. Let's reduce this system

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Given that there are three pivot and one free variables. Hence, we have nonzero solutions to $Ax = 0$ and v_1, v_2, v_3, v_4 are linearly dependent.
If we consider the matrix with columns v_1, v_2, v_3 the reduction steps are identical and they will lead to an identity matrix. This implies that there is no nonzero vector is the null space and hence v_1, v_2, v_3 are linearly independent.

10. problem 3.5.2

Solution:
we form a matrix with $v_1, \ldots, v_6$ as columns,

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

Reduction

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 3 pivot columns hence at most we can find 3 independent vectors. In this case v_1, v_2, v_3 are independent.

11. problem 3.5.9

Solution:
(a) We can have at most three independent rectors in $\mathbb{R}^3$ with 3 dimensions.

(b) If they span a two dimensional plane.

(c) zero vector cannot be in an independent set. In this case: $0\vec{v}_1 + c\vec{0} = \vec{0}$, for $c \neq 0$.

12. problem 3.5.23

Solution:
The second column of A is the sum of the other two columns. Same is true about the columns of U. Basis for $C(A)$: $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ Basis for $C(U)$: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$
Basis for $C\left(A^{\top}\right)$: $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ Basis for $C\left(U^{\top}\right)$: $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ The row space does not change after row reduction but the column space changes.