MATH 22: Homework 7 - Solutions

[Due Nov 3rd]

Written homework is intended to help students develop their communication and exposition skills through complete write-ups. While correctness of the solution is, of course, necessary, much of the grade for the problem is dependent on clear and appropriate exposition. Exposition shall be appropriate for the type and level of the problem. One principle we use is that exposition should be detailed around the main aspects of the problem, but terse exposition is appropriate for subsidiary parts of a problem.

1. problem 5.1.2

Solution: so if A is 3×3 with determinant -1, then $\det(\frac{1}{2}A) = \left(\frac{1}{2}\right)^3 \det A = \frac{1}{8}(-1) = -\frac{1}{8}$, by the row linearity of the determinant, since each of the three rows is multiplied by a half. Similarly, $\det(-A) = (-1)^3 \det(A) = -1(-1) = 1$. Next, since $\det(AB) = \det(A)\det(B)$, we have

$$\det(A^2) = \det(AA) = \det(A)\det(A) = (-1)(-1) = 1$$

and

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = (-1)\det(A^{-1})$$

so $\det(A^{-1}) = -1$

2. problem 5.1.7, just for the first matrix

Solution: Using our formula for the 2×2 determinant, this is $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta.$ This agrees with the notion of the determinant as the area of the parallelogram spanned by the columns of A, since this is the unit square rotated by θ, which doesn't change the area from one.

3. problem 5.1.9

Solution: XXXX

4. problem 5.1.13

Solution: Reducing these gives

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1(1)(1) = 1$$

and

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 3 & 3 & 3 \end{vmatrix} = -2 = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{vmatrix} =$$

$$= -2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 3/2 \\ 0 & -3 & -6 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 3/2 \\ 0 & 0 & -6+9/2 \end{vmatrix} = -2(1)(1)(-1.5) = 3$$

5. problem 5.2.3, except the "6 terms in det A"

Solution: By the cofactor formula, we have

$$\begin{vmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{vmatrix} = x \begin{vmatrix} 0 & x \\ 0 & x \end{vmatrix} - x \begin{vmatrix} 0 & x \\ 0 & x \end{vmatrix} + x \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = x0 - x0 + x0 = 0 + 0 + 0 = 0$$

Since those matrices are singular, because they have a column of zeros. All cofactors of row one are zero. The rank of A is two, since the first and second rows have pivots.

6. problem 5.2.12, except compare C^T to A^{-1}, not AC^T.

Solution: Since A is $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, it's cofactor matrix is

$$C = \begin{bmatrix} 2*2-(-1)(-1) & -(-1*2-0(-1)) & -1(-1)-0(2) \\ -(-1(2)-(-1)0) & 2(2)-0(0) & -(2(-1)-0(-1)) \\ -1(-1)-2(0) & -(2(-1)-(-1)0) & 2(2)-(-1)(-1) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

so $AC^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \det(A)I$, which is to be expected, as our formula is $A^{-1} = \frac{1}{\det A}C^T$, so $C^T = (\det A)A^{-1}$, and therefore $AC^T = A(\det AA^{-1}) = (\det A)I$.

7. If A is not invertible, but B is, can AB or BA ever be invertible? If so, find an example, if not, make an argument why this is the case.

Solution: No, since $\det(AB) = \det(A)\det(B) = 0\det(B) = 0$, since the determinant of a singular matrix is zero, and the determinant of a product is the product of determinants.

8. Is the transpose of an invertible matrix invertible? Will this always be the case?

Solution: Yes, since the determinant of a matrix equals the determinant of it's transpose, one will be invertible precisely when the other is.

Optional Practice:

5.1: 1, 8, 11, 15

5.2: 2, 4, 14, 15