MATH 22: Homework 8 - Solutions

[Due Nov 9th]

Written homework is intended to help students develop their communication and exposition skills through complete write-ups. While correctness of the solution is, of course, necessary, much of the grade for the problem is dependent on clear and appropriate exposition. Exposition shall be appropriate for the type and level of the problem. One principle we use is that exposition should be detailed around the main aspects of the problem, but terse exposition is appropriate for subsidiary parts of a problem.

1. problem 6.1.2

Solution: We solve this first for A, then note the general pattern (it is also fine to solve for $A+I$ explicitly. First we find the roots of the characteristic polynomial

$$\det(A-xI) = \begin{vmatrix} 1-x & 4 \\ 2 & 3-x \end{vmatrix} = (1-x)(3-x)-2(4) = 3-x-3x+x^2-8 = x^3-4x-5 = (x-5)(x+1)$$

which has roots at 5 and -1.

These are the eigenvalues, they correspond to eigenvectors in the nullspace of $A+1I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$ and $A-5I = \begin{vmatrix} -4 & 4 \\ 2 & -2 \end{vmatrix}$, which are $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively.

If $A\vec{v} = \lambda\vec{v}$, then $(A+I)\vec{v} = A\vec{v} + I\vec{v} = \lambda\vec{v} + \vec{v} = (\lambda+1)\vec{v}$, so $A+I$ has the same eigenvectors as A, but the eigenvalues are increased by one.

2. 6.1.5

Solution: The eigenvalues of a triangular matrix are its diagonal entries, as $A - xI$ will be a triangular matrix, so have determinant the product of the entries, which is the product of x minus each diagonal entry of A. Thus, the roots are the diagonal entries of A. In this case, they are 3 and 1 for A and also B.

The eigenvalues of $A+B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ are the roots of $(4-x)^2 - 1$. Setting this equal to zero and solving gives $(4-x)^2 = 1$, so $4 - x = \pm 1$, so $x = 3, -5$. This is NOT the sum of the eigenvalues of A and those of B.

3. 6.1.16

Solution: If $\lambda = 0$, the left hand side is $\det(A + 0I) = \det(A)$ while the right hand side is $(\lambda_1 - 0)...(\lambda_n - 0) = \lambda_1...\lambda_n$, so the two are equal.
In example 1, the determinant is $\begin{vmatrix} .8 & .3 \\ .2 & .7 \end{vmatrix} = .8(.7) - .2(.3) = .56 - .06 = 0.5 = 1/2 = 1(1/2) = \lambda_1\lambda_2$ as stated.

4. 6.2.2

Solution: Using the general form, $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ which indeed has the right eigenvalues / eigenvectors.

5. 6.2.5

Solution: If the eigenvectors of A are the columns of the identity matrix, the standard basis elements e_i, the vector of all zeros but with a 1 in component i. Then the ith column of A is $a_i = Ae_i = \lambda e_i$, so A is diagonal.
If S and A^{-1} are triangular, then $A = S\Lambda S^{-1}$ is a product of triangular matrices, so is triangular (Λ is diagonal, a special kind of triangular).

6. 6.2.14, find at least two entries you can change.

Solution: Since the rank of $A - 3I = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is one, and there are two columns, the nullspace will be one dimensional. Thus, we don't have enough eigenvectors to invert S in the diagonalization of A.
If change either one of the threes, say the top left to a, then A will still be triangular, so will have eigenvalues 3 and a, and the associated matrices will be

$$A - 3I = \begin{bmatrix} a-3 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A - aI = \begin{bmatrix} 0 & 1 \\ 0 & 3-a \end{bmatrix}$$

will have nullspaces spanned by $\begin{bmatrix} 1/(a-3) \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so we have enough eigenvectors to diagonalize A.

We could also change the top right entry to zero, then A would be $3I$ itself, which is already diagonal. Any nonzero entry in the top right would have the same issue as one.

Essentially, the textbook says distinct eigenvalues have distinct eigenvectors, so we just need to change an entry to make this happen. If we change the top right,

Lastly, if we change the bottom left entry, say to y, we could find the eigenvalues as the roots of $\det(A - xI) = (3-x)(3-x) - b(1) = x^3 - 6x + 9 - y$, and use the quadratic formula to solve for the value of y that make the roots distinct. This occurs when the quantity in the radical, $b^2 - 4ac = (-6)^2 - 4(1)(9-y) = 36 - 36 + 4y = 4y$ is nonzero, as then we are adding and subtracting something to get two different solutions. But that only happens when $4y = 0$, so $y = 0$, therefore any nonzero value in the bottom left would also make A diagonalizable.

7. Let $\vec{v}$ and $\vec{w}$ be eigenvectors of A. If $\vec{v} + \vec{w}$ is an eigenvector of A as well, then what must be true of the eigenvalues associated with $\vec{v}$ and $\vec{w}$?

Solution: Let the eigenvalues of $\vec{v}$ and $\vec{w}$ by λ and μ. Then $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = \lambda\vec{v} + \mu\vec{w} = a(\vec{v} + \vec{w})$ if and only if we can factor out $a = \lambda = \mu$, that is, the eigenvalues are the same.

8. The matrix P has eigenvalues 0, 1, and 1, corresponding to eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. The matrix Q has the same eigenvalues but corresponding to eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Use the diagonal formula $A = S\Lambda S^{-1}$ to find P and Q, what do you notice, and why might this be?

Solution: Using our formula, we find $P = Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which happens to be a projection matrix onto the xy plane. Because we have an eigenvalue $\lambda = 1$ with two associated eigenvectors, we could choose a different basis for the nullspace of $A - \lambda I$.
This example shows there are many possible descriptions of the same matrix in terms of eigenvalues and eigenvectors, because we choose a basis for each nullspace. Section 6.4 shows that symmetric matrices can have the basis of eigenvectors all unit vectors and mutually orthogonal.

9. If $\vec{v}$ is en eigenvector of A and of B, must it be an eigenvector of AB?

Solution: Yes, as $AB\vec{v} = A(\lambda_1)\vec{v} = \lambda_1 A\vec{v} = \lambda_1\lambda_2\vec{v}$.

Optional Practice:

6.1: 3, 6, 8, and 11

6.2: 3,4,9, and 12