NAME: ______________________________

SECTION: (Latifi 10:10) ☐ (Mintz 11:30) ☐

Math 22: Linear Algebra with Applications

Practice Midterm 2

October 27th, 2023

Instructions:

1. Please read all the instructions, and read and sign the statement at the bottom of this page.

2. Print your name *legibly* at the top of this page, and indicate your section by checking the appropriate box.

3. Some problems have multiple parts, do all of them. The different parts of a problem are not necessarily worth the same number of points.

4. Except on clearly indicated short answer problems, you must explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer. Make it clear and legible so we can follow it.

5. It is fine to leave your answer in a form such as $\ln(.02)$ or $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $e^{\ln(.02)}$ or $\cos(\pi)$ or $(3-2)$), you should simplify it.
Answers should *not* be rounded. If the answer is π then 3.14159 is wrong.
If the problem includes units, your answer should include units.

6. There are a few pages of scratch paper at the end of the exam. We *will not look* at these pages unless you write on a problem "Continued on page…"

7. Once finished, you will upload your exam to Gradescope. **You must assign pages to each question so that we can return your scores in a reasonable timeframe.** Once the exam is uploaded, give your physical copy to the instructors in case there is a problem with the upload. If you finish early, you should find the attending instructor before uploading so as not to distract others.

8. **Honor Code:** This exam is closed book. You may not consult notes, calculators (except as explicitly permitted for accessibility), phones, or any other external resource. Please turn off your devices until you have finished the exam.
It is a violation of the honor code to give or receive help on this exam.

STATEMENT: I have read these instructions, and I understand how the honor code applies to this exam.

SIGNATURE: ______________________________

1. (10 points) Which two matrices in the following list have the same nullspace?

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 3 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

Solution: The nullspace contains vectors with the same number of entries as there are columns in A. So $N(B)$ is is not in the same space an any other matrix, so can't be part of the pair. Each of the other matrices only has two vectors, so the nullspace will only be nontrivial if one is a multiple of the other, since this is the only way two vectors can be linearly dependent. We see this is not the case for A and B, so they both have the trivial nullspace $N(A) = N(B) = \{\vec{0}\}$. Indeed, D has the nullspace $N(D) = \left\{ a \begin{bmatrix} 2 \\ -1 \end{bmatrix}, a \text{ in } \mathbb{R} \right\}$

2. (10 points) You've been hired to consult a farm on their livestock selection. They're considering changing their set of animals, and currently have fifty goats and fifty sheep. Each animal has different grazing and housing requirements. The amounts of these per animal are summarized in the below table. The farm has 100 acres of grazing area, and a 750 square foot barn. They'd really prefer not to construct new housing for the animals, or purchase more land, so your task is to tell them what other sets of animals they could have.

Animal	Acres of Land	Square feet in barn
goat	1	10
cow	3	20
sheep	1	5

To make this easier for the farmers to interpret, you must also provide three concrete alternatives, different amounts of the livestock that they could provide for.

One way to approach this is to let g, c, and s be the number of goats, cow, and sheep the farm has, and write equations for each resource that many animals will use, and solve the resulting system.

Solution: Translating this into a system of equations, we have

$$g + 3c + s = 100$$

$$10g + 20c + 5s = 750$$

We can solve this by translating it into a matrix equation

$$A\vec{x} = \begin{bmatrix} 1 & 3 & 1 \\ 10 & 20 & 5 \end{bmatrix} \begin{bmatrix} g \\ c \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{b}$$

which we are looking for the complete solution to. We've already been given a particular solution, the current amounts of animals, so it suffices to find a basis for the nullspace, the special solutions. We can do this by first identifying the free columns, each will correspond to one of these.

$$\begin{bmatrix} 1 & 3 & 1 \\ 10 & 20 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 \\ 0 & -10 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \end{bmatrix}$$

We see there is one free column, the last, corresponding the vector $\begin{bmatrix} -1/2 \\ 1/2 \\ -1 \end{bmatrix}$ in the nullspace. We can interpret this as $A\vec{n} = -\frac{1}{2}\vec{a}_1 + \frac{1}{2}\vec{a}_2 - \vec{a}_2 = \vec{0}$, or $\vec{a}_2 = \vec{a}_1 + 2\vec{a}_2$, which is indeed correct. This can be interpreted as one cow use the same amount of these two resources as one goat and two sheep.
Thus the complete solution is

$$\left\{ \begin{bmatrix} 50 \\ 0 \\ 50 \end{bmatrix} + a \begin{bmatrix} -1/2 \\ 1/2 \\ -1 \end{bmatrix} : a \text{ in } \mathbb{R} \right\}$$

To make this more concrete, we can take $a = 2, 4, 6$ to get the solutions $\begin{bmatrix} 49 \\ 1 \\ 48 \end{bmatrix}, \begin{bmatrix} 48 \\ 2 \\ 46 \end{bmatrix}, \begin{bmatrix} 47 \\ 3 \\ 44 \end{bmatrix}$.
Another way to explain this to the farmer is "for every cow you want to get, you have to give up one goat and two sheep".

3. (10 points) Find the matrix P that projects onto the <u>row</u> space of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$. Note the projection matrix onto the <u>column</u> space of B is $P = B(B^T B)^{-1} B^T$.

It could be helpful to use the inverse formula

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Solution: Note we must first translate from the row space to the columns space to apply our formula (if we tried applying the formula to A we'd try to invert a singular matrix). Using the provided formula for $B = A^T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. We compute this in steps. First, we find

$$B^T B = AB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \to (B^T B)^{-1} = \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}$$

by our inverse formula, or Gaussian Elimination of the augmented matrix $[B^T B | I]$. Therefore

$$P = B(B^T B)^{-1} B^T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

4. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 2 & 3 & 1 \\ -1 & -2 & 1 & -4 \end{bmatrix}$$

(a) Find a basis for the null space of A.

(b) What is the dimension of the null space of A ? $\dim N(A) =$ _______

Solution: (a) To find the null space we apply vow reduction.

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 2 & 3 & 1 \\ -1 & -2 & 1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -4 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 3 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence x_1 and x_2 are pivot variables and x_3 and x_4 are free.
we have

$$\begin{cases} x_1 = 4x_3 - 3x_4 \\ x_2 = -\frac{3}{2}x_3 - \frac{1}{2}x_4 \end{cases}$$

$$N(A) = \left\{ \begin{bmatrix} 4x_3 - 3x_4 \\ -\frac{3}{2}x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} ; x_3, x_4 \in \mathbb{R} \right\}$$

$$= \left\{ x_3 \begin{bmatrix} 4 \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ \frac{-1}{2} \\ 0 \\ 1 \end{bmatrix} ; x_3, x_4 \in \mathbb{R} \right\}$$

(b) The null space has the basis

$$B = \left\{ \begin{bmatrix} 4 \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$$

hence $\dim N(A) = 2$ (which is the number of vectors in the basis = number of free variables after row reduction)

5. (5 points) The columns of a matrix A are linearly independent vectors if $N(A)$, the kernel of A is _______

Solution: The zero subspace.

6. (10 points) If A has m rows and n columns,
(a) What is the maximum possible value for the dimension of $N\left(A^T\right)$?
(b) If, in addition, we know that $\dim N(A^T) = 2$ what is the rank of A?

Solution:
(a) The maximum is m.

(b) The rank of A is $m - 2$.

7. (10 points) Let P be a 3 by 3 matrix associated with an orthogonal projection on a plane in $\mathbb{R}^3$. Given the following identities find P,

$$P\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$P\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

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Solution: Note that the projection matrix maps $u=\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $v=\begin{bmatrix}0\\0\\1\end{bmatrix}$ to themselves. That means both u and v are in the projection plane. Since the plane is a two dimensional subspace of $\mathbb{R}^3$ and u, v are independent then u, v must be a basis for the plane. Now, let us find projection matrix from u and v. Let

$$A=\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}$$

then we have $p=A\left(A^\top A\right)^{-1}A^\top$.

$$A^\top A=\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}=\begin{bmatrix}2&0\\0&1\end{bmatrix}$$

$$\left(A^\top A\right)^{-1}=\begin{bmatrix}\frac{1}{2}&0\\0&1\end{bmatrix}$$

$$\left(A^\top A\right)^{-1}A^\top=\begin{bmatrix}\frac{1}{2}&0\\0&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&0\\0&0&1\end{bmatrix}$$

$$\Rightarrow P=A\left(A^\top A\right)^{-1}A^\top=\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}&0\\0&0&1\end{bmatrix}$$

8. Let W be the subspace spanned by u, v,

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

(a) Find all vectors that are orthogonal to every vector in W.

(b) Do vectors found in part (a) form a subspace of $\mathbb{R}^4$? If yes, what is the dimension of that subspace?

Solution: (a) The set of vectors that are orthogonal to a subspace W are denoted usually by $W^{\perp}$ (the orthogonal complement of W). We form a matrix with u and v as rows,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

a vector $x \in \mathbb{R}^4$ is orthogonal to W if it's orthogonal to both basis vectors u and v. This means $W^{\perp} = N(A)$. To find the null space we reduce A to

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases}$$

then

$$N(A) = \left\{ \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} ; \quad x_3, x_4 \in \mathbb{R} \right\}$$

$$W^{\perp} = N(A) = \left\{ x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} ; x_3, x_4 \in \mathbb{R} \right\}$$

(b) Yes, $\omega^{\perp}$ is a subspace since $N(A)$ is a subspace. The rectors,

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $W^{\perp}$ hence $\dim W^{\perp} = 2$.