Homework 1 - Problem Set

In problem 1, 2 and 3 draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.

Problem 1: (Section 1.1 #1)
$y' = 3 - 2y$

Problem 2: (Section 1.1 # 3)
$y' = -1 - 2y$

Problem 3: (Section 1.1 #22)
$y' = -2 + t - y$

Problem 4: (Section 1.2 #7)
The field mouse population in Example 1 satisfies the differential equation

$$\frac{dp}{dt} = \frac{p}{2} - 450$$

(a) Find the time at which the population becomes extinct if $p(0) = 850$.
(b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.

Problem 5: (Section 1.2 #2)
Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature. Hence it satisfies the differential equation

$$\frac{du}{dt} = -k(u - T)$$

where is the constant ambient temperature and k is a positive constant. Suppose that $u(0) = u_0$ is the initial temperature,

(a) Find the temperature of the object at any time.
(b) Let τ be time at which the initial temperature difference $u_0 - T$ has been reduced by half. Find the relation between k and τ.

Problem 6: (Section 1.3 #11)
Determine the value of the r for which $y' + 2y = 0$ has the solution of the form e^{rt}. (optional question: what about $y'' + y' - 6y = 0$?)

Problem 7: A very important application of ODEs is to describe how probabilities change of time in random, or *stochastic*, processes called continuous time Markov Chains (CTMC). Perhaps the simplest example is a system (say the confirmation of a molecule) which can be in one of two states, 0 or 1, and switches between them at a constant rate h. Then the probabilities to be in states 0 and 1 (p_0 and p_1 respectively) obey the system of two differential equations:

$$\begin{aligned} \frac{d}{dt}p_0 &= -hp_0 + hp_1 \\ \frac{d}{dt}p_1 &= hp_0 - hp_1. \end{aligned} \tag{1}$$

These equations are subject to the conservation of probability constraint $p_0 + p_1 = 1$.

(a) Solve the differential equations (hint: rewrite the system as a single ODE) and describe the long-term behavior.

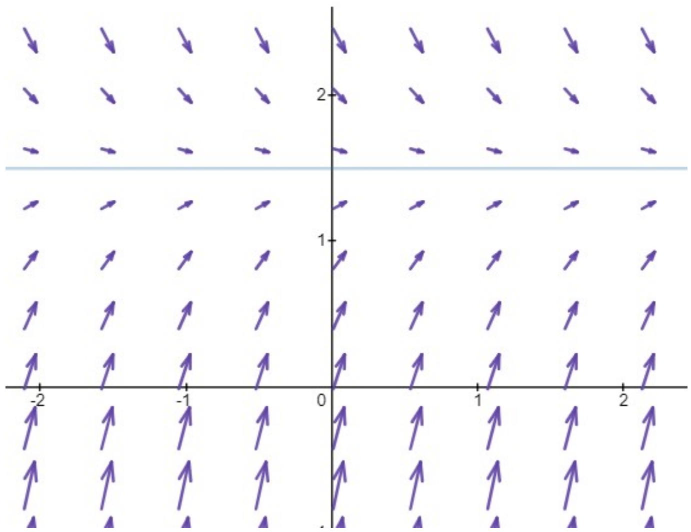
(b) For the more general Markov chain where the rates are different, the equations are

$$\begin{aligned} \frac{d}{dt}p_0 &= -hp_0 + wp_1 \\ \frac{d}{dt}p_1 &= hp_0 - wp_1. \end{aligned} \tag{2}$$

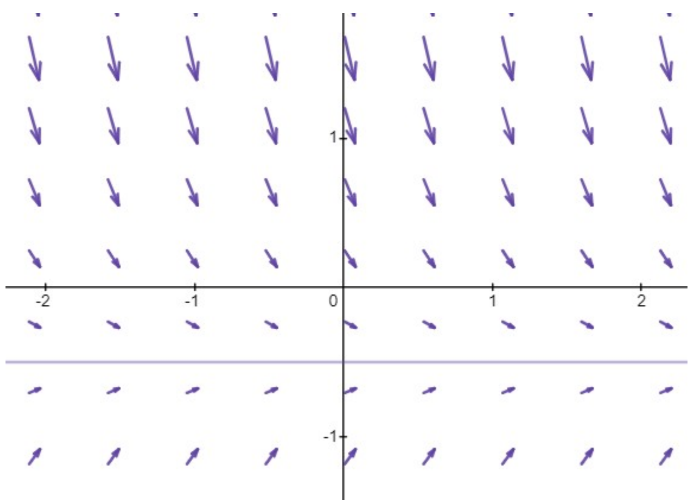
Without solving, describe the long-term behavior of this ODE.

Solutions

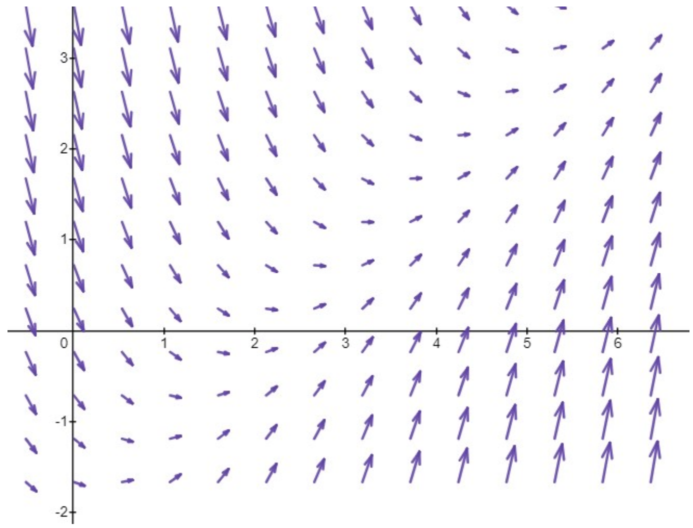
Problem 1 [Correct drawing of vector field +6 point] For any initial conditions the solutions tends towards 3/2. [Long-term behavior + 4 point]



Problem 2 [Correct drawing of vector field +6 point] For any initial conditions the solutions tend towards $-1/2$. [Long-term behavior +4 point]



Problem 3 [Correct drawing of vector field +6 point] For any initial conditions the solutions tend towards $y = x - 2$. [Long-term behavior +4 point]



Problem 4 Given $\frac{dp}{dt} = \frac{p}{2} - 450$ we have the solution

$$p(t) = 900 + Ce^{t/2}[+4 points]$$

(a) If $p(0) = 850$ we get $C = -50$ and $p(t) = 900 - 50e^{t/2}$ [+4 points]. Population becomes extinct when

$$900 - 50e^{t/2} = 0 \Rightarrow t = 2\ln(18)[+4 points]$$

(b) If $p(0) = p_0$ then $C = p_0 - 900 < 0$ and $p(t) = 900 + (p_0 - 900)e^{t/2}$ [+4 points], hence

$$900 + (p_0 - 900)e^{t/2} = 0 \Rightarrow t = 2\ln(\frac{900}{900 - p_0})[+4 points]$$

Problem 5 (a) Newton's law of heating can be stated as

$$\frac{du}{dt} = -k(u - T)$$

where $u(t)$ is the temperature at time t and T is the constant environment temperature. We can separate this equation as

$$k\,dt + \frac{1}{u - T}\,du = 0$$

leading to the solution $kt + \ln(u - T) = c_0$ or

$$u(t) = T + ce^{-kt}[+5 points]$$

Notice that $u(0) = T + c$ and $u(\infty) = T$. Now, given $u(0) = u_0$ we have

$$u(t) = T + (u_0 - T)e^{-kt}[5 points]$$

(b) Assuming $u(\tau) - T = (u_0 - T)/2$ we get

$$(u_0 - T)e^{-k\tau} = (u_0 - T)/2[+5 points]$$

which means $\tau = \ln(2)/k$ [+5 points].

Problem 6 From $(e^{rt})' + 2e^{rt} = re^{rt} + 2e^{rt} = 0$ we conclude $r = -2$ [+10 points].

Problem 7

(a) Let $p_0 = p$ therefore $p_1 = 1 - p$. The first equation (we could equivalently look at the second) becomes

$$\frac{d}{dt}p = -hp + h(1-p) = h - 2hp \qquad \text{[Identify equation +4]} \tag{3}$$

the solution is

$$p = Ce^{-2th} + \frac{1}{2}\text{[Solve +4]}. \tag{4}$$

In the long term p will tend towards 1/2 [Correct long-term behavior +4].

(b) For the more general situation,

$$\frac{d}{dt}p = w - (w+h)p \qquad \text{[Identify equation +4]} \tag{5}$$

We can see that if $p > w/(w-h)$ the probability to be in state 0 will decrease, while if $p > w/(w+h)$ the probability will increase. Thus p will tend towards $w/(w+h)$ [Identify equation +4].