Homework 1 - Problem Set

In problem 1, 2 and 3 draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.

Problem 1: (Section 1.1 #1)
$y' = 3 - 2y$

Problem 2: (Section 1.1 # 3)
$y' = -1 - 2y$

Problem 3: (Section 1.1 #22)
$y' = -2 + t - y$

Problem 4: (Section 1.2 #7)
The field mouse population in Example 1 satisfies the differential equation

$$\frac{dp}{dt} = \frac{p}{2} - 450$$

(a) Find the time at which the population becomes extinct if $p(0) = 850$.
(b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.

Problem 5: (Section 1.2 #2)
Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature. Hence it satisfies the differential equation

$$\frac{du}{dt} = -k(u - T)$$

where is the constant ambient temperature and k is a positive constant. Suppose that $u(0) = u_0$ is the initial temperature,

(a) Find the temperature of the object at any time.
(b) Let τ be time at which the initial temperature difference $u_0 - \tau$ has been reduced by half. Find the relation between k and τ.

Problem 6: (Section 1.3 #11)
Determine the value of the r for which $y' + 2y = 0$ has solution of the form e^{rt}. (optional question: what about $y'' + y' - 6y = 0$?)

Problem 7: A very important application of ODEs is to describe how probabilities change of time in random, or *stochastic*, processes called continuous time Markov Chains (CTMC). Perhaps the simplest example is a systems (say the confirmation of a molecule) which can be in one of two states, 0 or 1, and switches between them at a constant rate h. Then the probabilities to be in states 0 and 1 (p_0 and p_1 respectively) obey the system of two differential equations:

$$\begin{aligned} \frac{d}{dt}p_0 &= -hp_0 + hp_1 \\ \frac{d}{dt}p_1 &= hp_0 - hp_1. \end{aligned} \tag{1}$$

These equations are subject to the conservation of probability constraint $p_0 + p_1 = 1$.

(a) Solve the differential equations (hint: rewrite the system as a single ODE) and describe the long-term behavior.

(b) For the more general Markov chain where the rates are different, the equations are

$$\begin{aligned} \frac{d}{dt}p_0 &= -hp_0 + wp_1 \\ \frac{d}{dt}p_1 &= hp_0 - wp_1. \end{aligned} \tag{2}$$

Without solving, describe the long-term behavior of this ODE.