Homework 4 - Problem Set

(Numbered according to 9th edition)

Problem 1: Section 3.2 #4

Problem 2: Section 3.2 #9 (use Theorem 3.2.1)

Problem 3: Section 3.2 #25

Problem 4: Section 3.2 #29 (use Abel's Theorem 3.2.6)

Problem 5: Section 3.4 #6

Problem 6: Section 3.5 #7

Problem 7: Section 3.5 #16

Problem 8: Section 3.6 #1

Problem 9: Section 3.6 #17

3.2.4

$$W = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = x^2e^x$$

3.2.9

$$t(t-4)y'' + 3ty' + 4y = 2$$
$$y'' + \frac{3t}{t(t-4)}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

Given the initial values at $t_0 = 3$ we have a solution for $0 < t < 4$.

3.2.25

$y'' - 2y' + y = 0$
$y_1(t) = e^t, \quad y_2(t) = te^t$
$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$
$\quad = e^{2t} + te^{2^t} - te^{2^t} = e^{2t} \neq 0$

3.2.29

$$W = c\exp\left(-\int p(t)dt\right)$$
$$= c\exp\left(\int \frac{t(t+2)}{t^2}dt\right)$$

using

$$\int \frac{t^2+2t}{t^2}dt = \int 1 + \frac{2}{t}dt$$
$$= t + 2\ln(t)$$

We have $W = ct^2e^t$

3.4.6

$$y'' - 6y' + 9y = 0$$

Char. eq: $r^2 - 6r + 9 = 0 \Rightarrow r_1 = r_2 = -3$

$$y_1(t) = e^{-3t}$$
$$y_2(t) = te^{-3t}$$

general solution:

$$y(t) = c_1e^{-3t} + c_2te^{-3t}$$

3.5.7

$$2y'' + 3y' + y = t^2 + 3\sin t$$

1. Particular solution for t^2 term

$$2y'' + 3y' + y = t^2$$

Let $y = A_0t^2 + A_1t + A_2$, then

$$t^2 = 2(2A_0) + 3(2A_0t + A_1) + \left(A_0t^2 + A_1t + A_2\right)$$
$$\Rightarrow \begin{cases} A_0 = 1 \\ A_1 + 6A_0 = 0 \\ A_2 + 3A_1 + 4A_0 = 0 \end{cases} \Rightarrow \begin{cases} A_0 = 1 \\ A_1 = -6 \\ A_2 = 14 \end{cases}$$
$$\Rightarrow Y_1(t) = t^2 - 6t + 14$$

2. Particular solution for the term $3\sin(t)$:

$$2y'' + 3y' + y = 3\sin t$$

we let $y(t) = A\cos(t) + B\sin(t)$

$$2(-A\cos(t) - B\sin(t))$$
$$+ 3(-A\sin(t) + B\cos(t))$$
$$+ (A\cos(t) + B\sin(t)) = 3\sin(t)$$
$$\Rightarrow \begin{cases} -A + 3B = 0 \\ -3A - B = 3 \end{cases} \Rightarrow \begin{cases} A = -\frac{9}{10} \\ B = \frac{-3}{10} \end{cases}$$
$$\Rightarrow Y_2(t) = \frac{-9}{10}\cos(t) - \frac{3}{10}\sin(t)$$

3. complementary solution

$$2y'' + 3y' + y = 0$$

char. eq. $2r^2 + 3r + 1 = 0 \Rightarrow r_1 = -1, r_2 = -\frac{1}{2}$

$$\Rightarrow y_1(t) = e^{-t}, y_2(t) = e^{-\frac{t}{2}}$$

4. general solution

$$\begin{aligned} y(t) =& c_1 y_1 + c_2 y_2 + Y_1 + Y_2 \\ =& c_1 e^{-t} + c_2 e^{-\frac{t}{2}} + t^2 - 6t + 14 \\ &- \frac{3}{10}\sin(t) - \frac{9}{10}\cos(t) \end{aligned}$$

3.5.16
$y'' - 2y' - 3y = 3te^{2t}$
$y(0) = 1, y'(0) = 0$

1. complementary solution

$$\begin{aligned} & y'' - 2y' - 3y = 0 \\ \Rightarrow & r^2 - 2r\_3 = 0 \Rightarrow (r-3)(r+1) = 0 \\ \Rightarrow & r_1 = -1, r_2 = 3 \\ \Rightarrow & y_1(t) = e^{-t} \quad , y_2(t) = e^{3t} \end{aligned}$$

2. Particular solution $t^s(A_0 t + A_1)e^{2t}$. Since $r_1 \neq 2$ and $r_2 \neq 2$ we let $s = 0$,

$$\begin{aligned} Y(t) &= (A_0 t + A_1)e^{2t} \\ Y'(t) &= A_0 e^{2t} + 2A_0 te^{2t} + 2A_1 e^{2t} \\ &= (A_0 + 2A_1)e^{2t} + 2A_0 te^{2t} \\ Y''(t) &= 2(A_0 + 2A_1)e^{2t} + 2A_0 e^{2t} + 4A_0 te^{2t} \end{aligned}$$

$$\Rightarrow A_0 = -1 \quad , A_1 = -\frac{2}{3}$$

general solution:

$$y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{2}{3}e^{2t} - te^{2t}$$

using $y(0) = 1, y'(0) = 0$,

$$y(t) = e^{3t} + \frac{2}{3}e^{-t} - \frac{2}{3}e^{2t} - te^{2t}$$

3.6.1

$$y'' - 5y' + 6y = 2e^t$$

Homogeneous eq.

$$\begin{aligned} &y'' - 5y' + 6y = 0 \\ \Rightarrow &r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0 \\ \Rightarrow &r_1 = 2, r_2 = 3 \\ \Rightarrow &y_1(t) = e^{2t}, y_2(t) = e^{3t} \end{aligned}$$

$$W = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}$$

$$Y(t) = -e^{2+} \int \frac{(e^{3t})(2e^t)}{e^{5t}} dt$$
$$+ e^{3t} \int \frac{(e^{2t})(2e^t)}{e^{5t}} dt$$
$$Y(t) = 2\left(-e^{2t}\right)(-e^{-t}) + 2\left(e^{3t}\right)\left(-\frac{1}{2}e^{-2t}\right) = 2e^t - e^t = e^t$$

3.6.17

$$x^2y'' - 3xy' + 4y = x^2\ln(x)$$
$$y_1(x) = x^2, y_2(x) = x^2\ln(x)$$

Wronskian:

$$W = \begin{vmatrix} x^2 & x^2\ln(x) \\ 2x & 2x\ln(x) + x \end{vmatrix} = x^3$$
$$Y(x) = -x^2 \int \frac{(x^2\ln(x))(x^2\ln(x))}{x^3} dx$$
$$+ x^2\ln(x) \int \frac{(x^2)(x^2\ln(x))}{x^3} dx$$
$$\Rightarrow Y(x) = \frac{1}{6}x^2(\ln(x))^3$$