Homework 5 - Problem Set

(Numbered according to 9th edition)

Problem 1: Section 7.3 #1 (read Examples 1 and 2)

Problem 2: Section 7.3 #2

Problem 3: Section 7.3 #3

Problem 4: Section 7.3 #16

Problem 5: Section 7.3 #17

Problem 6: Section 7.3 #23

Problem 7: Section 7.5 #1

Problem 8: Section 7.5 #10

Problem 9: Section 7.5 #11

Solutions

7.3.1

$$\begin{cases} x_1 - x_3 = 0 \\ 3x_1 + x_2 + x_3 = 1 \\ -x_1 + x_2 + 2x_3 = 2 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 - x_3 = 0 \\ x_2 + 4x_3 = 1 \\ x_2 + x_3 = 2 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = x_3 \\ x_2 + 4x_3 = 1 \\ -3x_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = \frac{-1}{3} \\ x_2 = \frac{7}{3} \\ x_3 = -\frac{1}{3} \end{cases}$$

7.3.2

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_2 + x_3 = 1 \\ x_1 - x_2 + 2x_3 = 1 \end{cases}$$

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ -3x_2 + 3x_3 = -1 \\ -3x_2 + 3x_3 = 0 \end{cases}$$

the second and third equations are inconsistent hence no solution.

7.3.3

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ 2x_1 + x_2 + x_3 = 1 \\ x_1 - x_2 + 2x_3 = -1 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 - x_3 = 2 \\ -3x_2 + 3x_3 = -3 \\ -3x_2 + 3x_3 = -3 \end{cases}$$

second and third equations are the same hence we let $x_3 = c, x_2 = c + 1$ and

$$x_1 + 2(c + 1) - c = 2 \Rightarrow x_1 = -c$$

7.3.16

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$$

Eigenvalues $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(1-\lambda)+3=0 \rightarrow \lambda^2-6\lambda+8=0$$

$$\Rightarrow \lambda = \frac{6 \pm \sqrt{36-32}}{2} \rightarrow \lambda_1 = 2, \lambda_2 = 4$$

Eigenvectors

$$(A-2I)\xi^{(1)} = 0$$

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \xi^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$(A-4I)\xi^{(2)} = 0$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$$

$$\Rightarrow \xi^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

7.3.17

Eigenvalues

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = (\lambda-3)(\lambda+1)+8=0$$

$$\Rightarrow \lambda^2-2\lambda+5=0 \Rightarrow \lambda = \frac{2 \pm \sqrt{-16}}{2}$$

$$\Rightarrow \lambda_1 = 1+2i, \lambda_2 = 1-2i$$

Eigenvectors

$$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$$

$$\Rightarrow \xi^{(1)} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$$\begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix}\begin{bmatrix} \zeta_1 \\ \xi_2 \end{bmatrix} = 0$$

$$\Rightarrow \xi^{(2)} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

7.3.23

Eigenvalues

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(4-\lambda)(-1-\lambda) - 4 - 8 + 4(3-\lambda)$$
$$+ 2(1+\lambda) + 4(4-\lambda) = 0$$
$$\Rightarrow -\lambda^3 + 6\lambda^2 - 5\lambda - 12 + 18 - 6\lambda = 0$$
$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$
$$\Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

Eigenvectors

$$\xi^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$
$$\xi^{(2)} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$
$$\xi^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

7.5.1

$$x' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} x$$
$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0$$
$$\Rightarrow \lambda_1 = -1, \lambda_2 = 2$$
$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \xi^{(1)} = 0 \Rightarrow \xi^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \xi^{(2)} = 0 \to \xi^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

general solution

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

As $t \to \infty$ the solution tends to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ direction and for $t \to -\infty$ it tends toward $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

7.5.10

$$x' = \begin{bmatrix} 2 & 2+i \\ -1 & -1-i \end{bmatrix} x$$

$$\begin{vmatrix} 2-\lambda & 2+i \\ -1 & -1-i-\lambda \end{vmatrix} = \lambda^2 + (-1+i)\lambda - i = 0$$

$$\Rightarrow (\lambda-1)(\lambda+i) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -i$$

$$\begin{bmatrix} 1 & 2+i \\ -1 & -2-i \end{bmatrix} \xi^{(1)} = 0 \Rightarrow \xi^{(1)} = \begin{bmatrix} 2+i \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2+i & 2+i \\ -1 & -1 \end{bmatrix} \xi^{(2)} = 0 \Rightarrow \xi^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

general solution:

$$x(t) = c_1 e^{t} \begin{bmatrix} 2+i \\ -1 \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

7.5.11

$$x' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} x$$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{vmatrix} == -\lambda^3 + 4\lambda^2 + \lambda - 4$$

$$\lambda_1 = 4, \lambda_2 = 1, \lambda_3 = -1$$

$$\begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \xi^{(1)} = 0 \Rightarrow \xi^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\xi^{(2)} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\xi^{(3)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

general solution:

$$x(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$