Homework 6 - Problem Set

(Numbered according to 9th edition)

Problem 1: Section 7.7 #3

Problem 2: Section 7.7 #5

Problem 3: Section 7.7 #11 (read the discussion around equation (14) in page 415)

Problem 4: Section 7.7 #14

Problem 5: Section 7.7 #15 (a)

Solutions

7.7.3

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x$$

Eigenvalues

$$(2-\lambda)(-2-\lambda)+3=0$$
$$\lambda^2-1=0 \Rightarrow \lambda_1=1, \lambda_2=-1$$

Eigenvectors

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \xi^{(1)} = 0 \Rightarrow \xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \xi^{(2)} = 0 \Rightarrow \xi^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

general solution:

$$x(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Fundamental solution:

$$\Phi(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$$

Boundary condition $\Phi(0) = I$,

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ c_1 + 3c_2 = 0 \end{cases}$$
$$\Rightarrow \begin{cases} c_1 = \frac{3}{2} \\ c_2 = -\frac{1}{2} \end{cases}$$
$$\Rightarrow x(t) = \frac{3}{2} e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 + 3c_2 = 1 \end{cases}$$
$$\Rightarrow \begin{cases} C_1 = \frac{1}{2} \\ C_2 = -\frac{1}{2} \end{cases}$$
$$\Rightarrow x(+) = -\frac{1}{2} e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

hence

$$\Phi(t) = \begin{bmatrix} \frac{3}{2}e^t - \frac{1}{2}e^{-t} & -\frac{1}{2}e^t + \frac{1}{2}e^{-t} \\ \frac{3}{2}e^t - \frac{3}{2}e^{-t} & -\frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{bmatrix}$$

(that satisfies $\Phi(0) = I$)

7.7.5

$$x' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} x$$

Eigenvalues

$$(2-\lambda)(-2-\lambda)+5=0$$
$$\lambda^2+1=0$$
$$\lambda_1 = i, \lambda_2 = -i$$

Eigenvectors

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \xi^{(1)} = 0$$
$$\xi^{(1)} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix} \xi^{(2)} = 0$$
$$\xi^{(2)} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

general solution

$$x = c_1 e^{it} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Fundamental solution

$$\Phi(t) = \begin{bmatrix} (2+i)e^{it} & (2-i)e^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

Initial condition $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives

$$\begin{cases} (2+i)c_1 + (2-i)c_2 = 1 \\ c_1 + c_2 = 0 \end{cases}$$
$$\Rightarrow c_1 = -\frac{i}{2}, c_2 = \frac{i}{2}$$
$$\Rightarrow x(t) = -\frac{i}{2} e^{it} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} + \frac{i}{2} e^{-it} \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2\sin t + \cos t \\ \sin t \end{bmatrix}$$

Initial condition $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$\Rightarrow x(t) = \begin{bmatrix} -5\sin t \\ \cos t - 2\sin t \end{bmatrix}$$

Using these we obtain the fundamental solution

$$\Phi(t) = \begin{bmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{bmatrix}$$

7.7.11

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x, x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Using the fundamental solution in (3),

$$\begin{aligned} x(t) &= \Phi(t) \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \frac{7}{2} e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{3}{2} e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

7.7.14

$$\Phi(t) = \frac{1}{2} \begin{bmatrix} 3e^{t} - e^{-t} & -e^{t} + e^{-t} \\ 3e^{t} - 3e^{-t} & e^{t} + 3e^{-t} \end{bmatrix}$$

$$\Phi(t)\Phi(s) = \frac{1}{2} \begin{bmatrix} 3e^{t} - e^{-t} & -e^{t} + e^{-t} \\ 3e^{t} - 3e^{-t} & e^{t} + 3e^{-t} \end{bmatrix}$$

$$\frac{1}{2} \begin{bmatrix} 3e^{s} - e^{-s} & -e^{s} + e^{-s} \\ 3e^{s} - 3e^{-s} & e^{s} + 3e^{-s} \end{bmatrix}$$

let us check $(1,1)$ entry of the product

$$\begin{aligned} &\frac{1}{4}\left(3e^{t} - e^{-t}\right)\left(3e^{s} - e^{-s}\right) + \frac{1}{4}\left(-e^{t} + e^{-t}\right)\left(3e^{s} - 3e^{-s}\right) \\ &= \frac{1}{4}\left(9e^{t+s} - 3e^{t-s} - 3e^{-t+5} + e^{-(t+s)}\right) \\ &+ \frac{1}{4}\left(-3e^{t+s} + 3e^{t-5} + 3e^{-t+s} - 3e^{-(t+s)}\right) \end{aligned}$$

After cancelation this becomes

$$\frac{1}{2}\left(3e^{t+s} - e^{-(t+s)}\right)$$

which is the $(1,1)$ entry of $\Phi(t+s)$. other entries follow a similar calculation.

7.7.15 (a)

Consider initial value problem (I)

$$Z' = AZ, \quad Z(0) = \Phi(s)$$

where Φ is the fundamental solution of $\Phi' = A\Phi, \phi(0) = I$. (1) $Z(t) = \Phi(t)\Phi(s)$ is a solution to (I) since

$$\begin{aligned} Z'(t) &= \Phi'(t)\Phi(s) \\ &= A\Phi(t)\Phi(s) \\ &= AZ(t) \end{aligned}$$

and $Z(0) = \Phi(0)\Phi(s) = \Phi(s)$

(2) $Z(t) = \Phi(t+s)$ is a solution to (I),

$$\begin{aligned} Z'(t) &= \Phi'(t+s) \\ &= A\Phi(t+s) \\ &= AZ(t) \end{aligned}$$

and $Z(0) = \Phi(0+s)$.

Using (1) and (2) and the fact that (I) has a unique solution we conclude that $\Phi(t+s) = \Phi(t)\Phi(s)$, i.e.

$$e^{(t+s)A} = e^{tA}e^{sA}$$