Homework 7 - Problem Set

(Numbered according to 9th edition)

Problem 1: Section 7.8 #2

Problem 2: Section 7.8 #18 (only a,b,c)

Problem 3: Section 7.9 #1

Problem 4: Section 7.9 #13

Solutions

7.8.2

$x' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} x$

Eigenvalues:

$$\begin{vmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{vmatrix} = \lambda^2 - 16 + 16 = 0$$
$$\Rightarrow \lambda = 0$$

Eigenvector:

$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \xi = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

generalized eigenvector:

$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \eta = \xi$$
$$\Rightarrow \begin{cases} 4\eta_1 - 2\eta_2 = 1 \\ 8\eta_1 - 4\eta_2 = 2 \end{cases}$$

any vector independent of ζ works. Let $\eta = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$.

general solution:

$$\begin{aligned} x(t) &= c_1\xi + c_2(t\xi + \eta) \\ &= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{4} c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

7.8.18

$x' = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} x$

(a) Verify that $A\xi^{(1)} = \xi^{(1)}$ and $A\xi^{(2)} = \xi^{(2)}$. Given

$$A - I = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix}$$

the rank of $A - I$ is two. Hence

$$(A - I)x = 0$$

has two independent solutions.

(b) Assuming we have a solution of the form $x = \xi + e^t + \eta e^t$, we obtain

$$\left(e^t + te^t\right)\xi + e^t\eta = A\left(+e^t\xi + e^t\eta\right)$$

which implies

$$\begin{aligned} \xi &= A\xi \\ \xi + \eta &= A\eta \end{aligned}$$

or equivalently,

$$\begin{aligned} (A - I)\xi &= 0 \\ (A - I)\eta &= \xi \end{aligned}$$

(c) We know that $(A - I)x = 0$ can have at most two linearly independent solutions. Any linear combination also gives an eigenvector

$$\begin{aligned} (A - I)\left(c_1\xi^{(1)} + c_2\xi^{(2)}\right) = &c_1(A - I)\xi^{(1)} \\ &+ c_2(A - I)\xi^{(2)} \\ =&c_1\xi^{(1)} + c_2\xi^{(2)} \end{aligned}$$

7.9.1

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x + \begin{bmatrix} e^t \\ t \end{bmatrix}$$

Eigenvalues of $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 3 = 0$$

$$\Rightarrow \lambda_1 = -1, \lambda_2 = 1$$

Eigenvectors:

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \xi^{(1)} = 0 \Rightarrow \xi^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \xi^{(2)} = 0 \Rightarrow \xi^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Transformation matrix:

$$T = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$$

where

$$T^{-1} \cdot AT = D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

In terms of y with $x = T_y$ our differential equation becomes

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + T^{-1}\begin{bmatrix} e^t \\ t \end{bmatrix}$$

where

$$\begin{aligned} T^{-1}\begin{bmatrix} e^t \\ t \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} e^t \\ t \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -e^t + t \\ 3e^t - t \end{bmatrix} \end{aligned}$$

this gives us two first-order equations:

$$\begin{aligned} \frac{dy_1}{dt} &= -y_1 + \frac{1}{2}(-e^t + t) \\ \frac{dy_2}{dt} &= y_1 + \frac{1}{2}(3e^t - t) \end{aligned}$$

We can solve this using the method of integrating factors, for instance,

$$\begin{aligned} y' &= y + \frac{1}{2}(3e^t - t) \\ (e^{-t}y) &= \frac{1}{2}(3 - te^{-t}) \\ e^{-t}y &= \frac{3}{2}t - \frac{1}{2}\int te^{-t}dt \\ &= \frac{3}{2}t + \frac{1}{2}te^{-t} + \frac{1}{2}e^{-t} \end{aligned}$$

Using the method of integrating factors,

$$\begin{aligned} y_1(t) &= \frac{-1}{4}e^t + \frac{1}{2}t - \frac{1}{2} \\ y_2(t) &= \frac{3}{2}te^t + \frac{t}{2} + \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} =& \frac{1}{4}e^t\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \frac{3}{2}te^t\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &+ \frac{1}{2}t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

In terms of $x = Ty$,

$$x = \frac{1}{4}e^t\begin{bmatrix} -1 \\ -3 \end{bmatrix} + \frac{3}{2}te^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

this a particular solution. The general solution is

$$x(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$+ \frac{3}{2} t e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{4} e^t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$+ t \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

7.9.13

[only graded for completion]

$$x' = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix} x + I(t) \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

Eigenvalues and eigenvectors

$$\lambda_1 = -\frac{1}{2} + \frac{i}{2}, \lambda_2 = -\frac{1}{2} - \frac{i}{2}$$
$$\xi^{(1)} = \begin{bmatrix} i \\ 4 \end{bmatrix}, \xi^{(2)} = \begin{bmatrix} -i \\ 4 \end{bmatrix}$$

change of coordinate transformation

$$T = \begin{bmatrix} i & -i \\ 4 & 4 \end{bmatrix}, T^{-1} = \frac{1}{8} \begin{bmatrix} -4i & 1 \\ 4i & 1 \end{bmatrix}$$

Equations in y coordinate, $x = T_y$

$$y_1' = \left(\frac{-1}{2} + \frac{i}{2}\right) y_1 - \frac{1}{4} i e^{-\frac{t}{2}}$$
$$y_2' = \left(-\frac{1}{2} - \frac{i}{2}\right) y_2 + \frac{1}{4} i e^{-t/2}$$

with Solution

$$y_1(t) = \frac{1}{2} e^{-t/2}, y_2(t) = \frac{1}{2} e^{-t/2}$$

the general solution in terms of x,

$$x(t) = c_1 e^{(-\frac{1}{2} + \frac{i}{2})t} \begin{bmatrix} i \\ 4 \end{bmatrix}$$
$$+ c_2 e^{(-\frac{1}{2} - \frac{i}{2})t} \begin{bmatrix} -i \\ 4 \end{bmatrix} + 4e^{-t/2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

fundamental solution

$$\psi(t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos\left(\frac{1}{2}t\right) & \sin\left(\frac{1}{2}t\right) \\ 4\sin\left(\frac{1}{2}t\right) & -4\cos\left(\frac{1}{2}t\right) \end{bmatrix}$$

Solution to the initial value problem

$$x(t) = e^{-t/2}\begin{bmatrix} \sin\left(\frac{1}{2}t\right) \\ 4 - 4\cos\left(\frac{1}{2}t\right) \end{bmatrix}$$