## Grading

Problems will be graded for accuracy and clarity of both mathematics and writing. If a problem involves an established solution method, such as integrating factors or separation of variables, you do not need to derive the method, but you need to be clear what method you are using and how you are using it. For example, if you use an integrating factor you must right down what the integrating factor is.

## Problems

1. Find the general solution to the ODE

$$x'' + 2x' + x = e^{2t}$$

2. Write down an equation describing a mass-spring system that will undergo decaying oscillations with a period of 2 minutes.

3. When solving second-order constant coefficient equations with repeated roots, the fundamental set of solutions has the form

$$y_1(t) = e^{rt}, \quad y_2(t) = te^{rt}$$

Show that these are able to satisfy any initial conditions (that is, they form a fundamental set).

4. Find the general solution to

$$\frac{d}{dt}\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} + e^{t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

5. Find the Laplace transform of $f(t) = t\cos(at)$.

6. Consider the system

$$\frac{d}{dt}x = x - y$$
$$\frac{d}{dt}y = xy^2$$

This system has a single fixed point at $(x, y) = (0, 0)$.

(a) Sketch a direction field. What does it suggest about the stability of $(x, y) = (0, 0)$ ?

(b) What does the Jacobian matrix tell us about the stability of the critical point?

7. Consider the linear system
$$\frac{d}{dt}\mathbf{x} = A\mathbf{x}, \quad A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

   (a) Find the eigenvectors and eigenvalues of $A$

   (b) Sketch a phase portrait of the system.

8. Consider the autonomous system

$$dx/dt = y, \quad dy/dt = x + 2x^3$$

(a) Show that the critical point $(0,0)$ is a saddle point.

(b) Sketch the trajectories for the corresponding linear system by integrating the equation for $dy/dx$. Show from the parametric form of the solution that the only trajectory on which $x \to 0, y \to 0$ as $t \to \infty$ is $y = -x$.

(c) Determine the trajectories for the nonlinear system by integrating the equation for $dy/dx$. Sketch the trajectories for the nonlinear system that correspond to $y = -x$ and $y = x$ for the linear system.

**Solutions**

# P1

Solution: The solution is a combination of the general solution to the homogeneous problem and the particular solution to the non-homogeneous problem. For the former the characteristic equation is

$$r^2 + 2r + 1 = 0 \Longrightarrow r = -1 \pm \frac{1}{2}\sqrt{4-4} = -1$$

Therefore we have repeated roots and the general solution is

$$\phi(t) = c_1 e^{-t} + c_2 t e^{-t}$$

Next, we try plugging in

$$Y(t) = Ae^{2t}$$

which yields

$$A4e^{2t} + 4Ae^{2t} + Ae^{2t} = e^{2t} \Longrightarrow 9Ae^{2t} = e^{2t} \Longrightarrow A = \frac{1}{9}$$

thus the general solution is

$$x(t) = \phi(t) + Y(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{9}e^{2t}$$

# P2

Solution: The roots of the characteristic equation should look like

$$\lambda = -r \pm i\omega$$

where $\omega = \frac{2\pi}{T}$ and $T = 2$ (minutes). we let $r = 1$ and hence

$$\lambda = -1 \pm i\pi$$

The characteristic equation is

$$\begin{aligned}(\lambda - \lambda_1)(\lambda - \lambda_2) &= (\lambda + 1 + i\pi)(\lambda + 1 - i\pi) \\ &= \lambda^2 + 2\lambda + 1 + \pi\end{aligned}$$

The 2nd order equation is

$$x'' + 2x' + (1+\pi)x = 0$$

# P3

Solution: In order to do this we need to check the Wronskian is not zero.

$$y_1(t)y_2'(t) - y_2(t)y_1'(t) = e^{rt}\left(rte^{rt} + e^{rt}\right) - rte^{2rt} = e^{2rt} \neq 0.$$

Thus the solutions form a fundamental set.

## P4

Solution:

## P5

Solution: This practice uses the Laplace transform of a step function (hence will not be on the test but the general idea applies)

$$\mathcal{L}\left\{u_{2\pi}(t)\right\} = \frac{e^{2\pi s}}{s}$$

so the Laplace transformed equation is (letting $Y(s) = \mathcal{L}y(t)$ )

$$s^2Y(s) - sy(0) - y'(0) + 2Y(s) = \left(s^2+2\right)Y(s) - s = 2\frac{e^{-2\pi s}}{s}$$

Solving for $Y(s)$ we get

$$Y(s) = \frac{2\frac{e^{2\pi s}}{s} + s}{s^2+2} = 2\frac{e^{-2\pi s}}{s\left(s^2+2\right)} + \frac{s}{s^2+2}$$

Using partial factions

$$\frac{1}{s\left(s^2+2\right)} = \frac{a}{s} + \frac{b}{s^2+2} = \frac{a\left(s^2+2\right)+bs}{s\left(s^2+2\right)}$$

implying $a\left(s^2+2\right)+bs = 1$ and hence $a = 1/2$ and $b = -1/2s$. Recalling the Laplace transform of $1/s$ is 1 and $s/\left(s^2+a^2\right)$ is $\cos(at)$ (this are in the table) we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s\left(s^2+2\right)}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} = \frac{1}{2} - \cos(\sqrt{2}t)$$

Next we use that multiplication by $e^{2\pi s}$ corresponds to multiplication by the step function in the inverse Laplace transform, so the final solutions is

$$Y(s) = 2u_{2\pi}\left(\frac{1}{2} + \cos(\sqrt{2}t)\right) + \cos(\sqrt{2}t)$$

## P6

Solution: (a) The direction field seems to suggest the fixed point at $(0,0)$ is unstable.

(b) Linear approximation near the origin. The Jacobian at $(0,0)$ is

$$J(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The phase portrait of this linear system consists of horizontal trajectories pointing away from the line $y = x$ (one may call this unstable but our book does not have a solid terminology for this degenerate case). The determinant is zero and the trace is one. Looking at the stability diagram (with det and trace axis), this means the linearized system happens to be on the boundary of two regions (above and below the horizontal axis). Therefore, small perturbations or errors can turn this into an unstable node or a saddle point. Hence we cannot draw any conclusion from the linearized system about the original one.

## P7

Solution: (a) The eigenvalues of $A$ are

$$r = \frac{3}{2} \pm \frac{1}{2}\sqrt{9 - 4 \times 2} = \frac{3}{2} \pm \frac{1}{2}$$

so $r_1 = 2$ and $r_2 = 1$. The eigenvector that goes with $r_1$ satisfies

$$v_1^{(1)} + 2v_2^{(1)} = 2v_1^{(1)} \Longrightarrow 2v_2^{(1)} = v_1^{(1)}$$

so an eigenvector is $\mathbf{v}^{(1)} = (2, 1)^T$. For the eigenvector that goes with $r_2$

$$v_1^{(1)} + v_2^{(1)} = v_1^{(1)} \Longrightarrow v_2^{(1)} = 0$$

so $\mathbf{v}^{(1)} = (1, 0)^T = \mathbf{e}_1$. (b) Based on the results above we have an unstable node and the large eigenvalue corresponds to the direction $(2, 1)$.

## P8

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = x + 2x^3 \end{cases}$$

(a) Jacobian at $(0, 0)$,

$$J(x,y) = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 + 6x^2 & 0 \end{bmatrix}$$
$$\Rightarrow J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Near the origin:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Eigenvalues:

$$\det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 \Rightarrow \lambda_1 = 1, \lambda_2 = -1$$

since eigenvalues are real opposite sign this is a saddle point.
(b)
Linear system near $(0, 0)$

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = x \end{cases}$$

trajectories:

$$\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{x}{y} \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$
$$\Rightarrow ydy - xdx = 0$$
$$\Rightarrow y^2 - x^2 = c$$

The only constant $c$ such that $x^2 - y^2 = c$ goes through the origin is $c = 0$. Hence, $x^2 = y^2$ and $y = \pm x$.

the parametric form of the solution near $(0, 0)$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \lambda_1 = -1, \lambda_2 = 1$$

Eigenvectors:

$$\xi^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \xi^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

parametric solution:

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

as $t \to \infty$ we have $x(t) \to 0$ and $y(t) \to 0$ only when $c_2 = 0$ and $c_1 \neq 0$ which happens along $y = -x$.

(c)

Trajectories of the non-linear system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x + 2x^3 \\ \frac{dy}{dx} = \frac{x + 2x^3}{y} &\Rightarrow ydy - \left(x + 2x^3\right) dx = 0 \\ &\Rightarrow y^2 - x^2 - x^4 = c \end{aligned}$$

If such a curve goes through the origin then $c = 0$ and

$$y^2 - x^2 - x^4 = 0$$