Grading

Problems will be graded for accuracy and clarity of both mathematics and writing. If a problem involves an established solution method, such as integrating factors or separation of variables, you do not need to derive the method, but you need to be clear what method you are using and how you are using it. For example, if you use an integrating factor you must right down what the integrating factor is.

Problems

1. Find the general solution to the ODE

$$x'' + 2x' + x = e^{2t}$$

2. Write down an equation describing a mass-spring system that will undergo decaying oscillations with a period of 2 minutes.

3. When solving second-order constant coefficient equations with repeated roots, the fundamental set of solutions has the form

$$y_1(t) = e^{rt}, \quad y_2(t) = te^{rt}$$

Show that these are able to satisfy any initial conditions (that is, they form a fundamental set).

4. Find the general solution to

$$\frac{d}{dt}\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} + e^t \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

5. Find the Laplace transform of $f(t) = t\cos(at)$.

6. Consider the system

$$\frac{d}{dt}x = x - y$$
$$\frac{d}{dt}y = xy^2$$

This system has a single fixed point at $(x, y) = (0, 0)$.

(a) Sketch a direction field. What does it suggest about the stability of $(x, y) = (0, 0)$?

(b) What does the Jacobian matrix tell us about the stability of the critical point?

7. Consider the linear system

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x}, \quad A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

(a) Find the eigenvectors and eigenvalues of A

(b) Sketch a phase portrait of the system.

8. Consider the autonomous system

$$dx/dt = y, \quad dy/dt = x + 2x^3$$

(a) Show that the critical point $(0, 0)$ is a saddle point.

(b) Sketch the trajectories for the corresponding linear system by integrating the equation for dy/dx. Show from the parametric form of the solution that the only trajectory on which $x \to 0, y \to 0$ as $t \to \infty$ is $y = -x$.

(c) Determine the trajectories for the nonlinear system by integrating the equation for dy/dx. Sketch the trajectories for the nonlinear system that correspond to $y = -x$ and $y = x$ for the linear system.