- motivation

↳ DFQs describe change
↳ change = derivative
↳ what is the rate of change of a variable given the state of the "universe" right now?
↳ Harry Potter adjucting the direction of his broom (the slope) based on his position in the xy-plane

- general form of a DFQ:

$\frac{dy}{dx}$ = some expression in terms of x and y

↳ here:

x → independent variable (e.g. time of the day)
y → dependent variable (e.g. your mood)

$\Delta x \to dx$
$\Delta y \to dy$
infinite decimal

$$\frac{\Delta y}{\Delta x} = f(x,y) \quad \to \quad \Delta y = f(x,y) \cdot \Delta x$$

↳ tells you how the change in y is affected by the change in x

- examples

① do you know any function $y(x)$ s.t. $\frac{dy}{dx} = 2$?

$y(x) = 2x + c$ → ∞ - many such functions

↳ what if $y(0) = 5$? then: $5 = 2 \cdot 0 + c \Rightarrow c = 5$. so:

$y(x) = 2x + 5$

② do you know any function $y(x)$ s.t. $\frac{dy}{dx} = 3x$?

$y(x) = \frac{3}{2} x^2 + c$ → ∞ - many such functions

③ do you know any function $y(x)$ s.t. $\frac{dy}{dx} = y$?

$y(x) = c \cdot e^x$ or $y(x) = e^{x+c}$ → ∞ - many solutions

↳ what if $y(0) = 3$? then: $3 = c \cdot e^0 \Rightarrow c = 3$. thus:

$y(x) = 3 \cdot e^x$.

④ do you know any function $y(x)$ s.t. $\frac{dy}{dx} = -2y$?

$y(x) = ce^{-2x}$ → ∞ - many such functions

- terminology

↳ $\frac{dy}{dx} = 2$ → DFQ

↳ $y(x) = 2x + c$ → general solution

↳ $y(0) = 5$ → boundary condition

↳ $y(x) = 2x+5$ → particular solution

① $\frac{dy}{dx} = -y$ ↳ boundary condition: $y(0) = 1$:

$y = c \cdot e^{-x}$. $\downarrow$

$y = 1 \cdot e^{-x}$

- terminology

→ classification of DFQs based on order:

1) first-order DFQ → the highest derivative is the 1st

↳ $y' = y$

2) second-order DFQ → the highest derivative is the 2nd

↳ $y'' = y' - y$

3) higher-order DFQ → $\frac{d^n}{dx^n}[y] = f(x, y, \frac{dy}{dx}, \ldots, \frac{d^ny}{dx^n})$

$F(x, y, \frac{dy}{dx}, \ldots, \frac{d^n}{dx^n}[y])$

↳ recall the meaning of the second derivative:

↳ second derivative measures the concavity of a function:

i) if $y'' < 0$: y is concave down ⌒ at that point

ii) if $y'' > 0$: y is concave up ◡ at that point

iii) if $y'' = 0$: $y = mx + b$ ⁄

→ ordinary vs. partial DFQs:

1) ordinary DFQs → usually 1 independent variable x

2) partial DFQs → involving partial derivatives

→ system of DFQs:

↳ multiple (possibly related) DFQs and unknown functions.

$$\left.\begin{aligned}\frac{dy}{dt} &= 3x + 4y \\ \frac{dx}{dt} &= x - y\end{aligned}\right\}$$

the solution functions x(t) and y(t) must both satisfy the equations

→ linear vs. non-linear DFQs:

1) linear DFQs → no powers in y or y'

↳ $f_0(x) \cdot y + f_1(x) \cdot y' + \dots + f_n(x) \cdot y^{(n)} = g(x)$

2) non-linear DFQs → $\frac{dy}{dx} + \sin(y) = 0$

↳ $(y')^2 + y^2 = 1$ $y = \sin(x)$ or $y = \cos(x)$

↳ note: $e^x \cdot y' + \sin(x) \cdot y'' = 0$ is linear

- modeling

① population of field mice w/o the predators.
hypothesis: mouse population growth is proportional to the size of the current population: $\frac{dP}{dt} \propto P(t)$

$\frac{dP}{dt} = k \cdot P(t)$, where $P(t)$ → population at time t
k → growth rate of the population
↳ how fast P grows.
↳ e.g. 0.5/month , −0.2/month

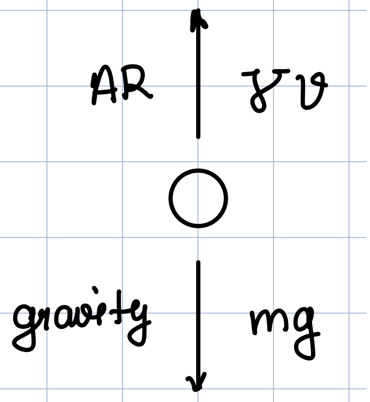
↳ general solution to this DFQ: $P(t) = C \cdot e^{kt}$
↳ exponential growth

② Newton's second law: $F = m \cdot a$ or $F = m \cdot \frac{dv}{dt}$

↳ for a falling object: F = gravity + air resistance

$$m \cdot \frac{dv}{dt} = m \cdot g - \gamma \cdot v$$

units: $kg \cdot \frac{m}{s^2}$ $\quad kg \cdot \frac{m}{s^2} - \frac{kg}{s} \cdot \frac{m}{s}$



↳ recall: if $y'(t) = g(t)$, to get $y(t)$:

$$y(t) = \int g(t)\,dt + c \quad \rightarrow \text{FTOC}$$

→ first-order linear ODEs: $\frac{dy}{dt} + p(t)\cdot y = g(t)$

↳ suppose $y' + py = g$. ⊛

↳ note: $(\int p)' = p$

↳ let $\mu = e^{\int p}$. observation: $(e^{\int p})' = p\cdot e^{\int p}$.

thus: $(e^{\int p}\cdot y)' = \underline{p\cdot e^{\int p}\cdot y + e^{\int p}\cdot y'}$.

↳ if we multiply ⊛ with $e^{\int p}$:

$\underline{e^{\int p}\cdot y' + p\cdot e^{\int p}\, y} = e^{\int p}\cdot g$ and notice the identity above.

↳ thus: $(e^{\int p}\cdot y)' = e^{\int p}\cdot g$ and integrating:

$$e^{\int p}\cdot y = \int e^{\int p}\cdot g + c$$

↳ solution to first-order ODE:

$y(t) = \frac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)\cdot g(s)\,ds$, where $\mu(t) = e^{\int p(t)\,dt}$

① $\frac{dy}{dt} - 2y = 4 - t \qquad p(t) = -2 \qquad \int p(t)\,dt = -2t \ (+\ c)$

↳ integrating factor: $\mu(t) = e^{-2t}$

↳ multiply the DFQ by $\mu(t)$:

$$e^{-2t}\frac{dy}{dt} - 2e^{-2t}y = e^{-2t}(4-t)$$

↳ replace the left-hand side with the identity $(e^{\int p}\cdot y)'$

$$\frac{d}{dt}\left(e^{-2t}y\right) = e^{-2t}(4-t)$$

↳ integrate w/ respect to t:

$$e^{-2t}y = \int_{t_0}^{t} e^{-2s}(4-s)\,ds + C \qquad \begin{array}{ll} f(s) = 4-s & g(s) = -\frac{1}{2}e^{-2s} \\ f'(s) = -1 & g'(s) = e^{-2s} \end{array}$$

by IBP: $$= (4-s)\cdot\frac{-1}{2}e^{-2s} - \int +1\cdot\frac{1}{2}e^{-2s}\,ds + C$$

$$= \frac{1}{2}(s-4)e^{-2s} + \frac{1}{4}\cdot e^{-2s} + c\,\Big]_{s=0}^{s=t} + C$$

$$= \frac{1}{2}(t-4)e^{-2t} + \frac{1}{4}\cdot e^{-2t} - \frac{1}{2}(-4) - \frac{1}{4} + C$$

$$e^{-2t}y = \frac{1}{2}te^{-2t} - 2e^{-2t} + \frac{1}{4}e^{-2t} + \underbrace{\frac{7}{4} + C}_{C} \ \Big|\cdot e^{2t}$$

$$y = \frac{1}{2}t - \frac{7}{4} + C\cdot e^{2t}$$ → general solution

② $y' + 3y = t + e^{-2t}$ $\qquad p(t) = 3 \implies \int p(t)\,dt = 3t\ (+c)$

↳ integrating factore : $\mu(t) = e^{3t}$

↳ multiply the DFQ by $\mu(t)$:

$$e^{3t}\frac{dy}{dt} + 3e^{3t}y = e^{3t}t + e^{t}$$

↳ replace the left-hand side with the identity $(e^{\int p}\cdot y)'$

$$\frac{d}{dt}\left(e^{3t}y\right) = e^{3t}t + e^{t}$$

$f(s) = s \qquad g(s) = \frac{1}{3}e^{3s}$

$f'(s) = 1 \qquad g'(s) = e^{3s}$

↳ integrate w/ respect to t:

$$e^{3t}\cdot y = \int_{t_0}^{t} e^{3s}\cdot s\,ds + \int_{t_0}^{t} e^{s}\,ds + c$$

$$= s\cdot\frac{1}{3}e^{3s} - \int 1\cdot\frac{1}{3}e^{3s}\,ds + e^{s} + c$$

$$= \frac{1}{3}se^{3s} - \frac{1}{3}\int e^{3s}\,ds + e^{s} + c$$

$$= \frac{1}{3}se^{3s} - \frac{1}{9}e^{3s} + e^{s} + C\Big]_{s=0}^{s=t} + c$$

$$e^{3t}\cdot y = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^{t} + \underbrace{\frac{1}{9} - 1 + c}_{C}\ \Big|\cdot e^{-3t}$$

$$y = \frac{1}{3}t - \frac{1}{9} + e^{-2t} + Ce^{-3t}$$ → general solution

③ $y' + \frac{2}{t} y = \frac{\cos(t)}{t^2}$ $\quad p(t) = \frac{2}{t}$ $\quad \int \frac{2}{t}\, dt = 2\ln(|t|) + c$

↳ integrating factor: $\mu(t) = e^{2\ln(|t|)} = t^2$

↳ multiply the DFQ by $\mu(t)$:

$$t^2 \frac{dy}{dt} + 2t\, y = \cos(t)$$

↳ replace the left-hand side with the identity $(e^{\int p} \cdot y)'$

$$\frac{d}{dt}(t^2 \cdot y) = \cos(t)$$

↳ integrate w/ respect to t:

$$t^2 \cdot y = \int \cos(t)\, dt + c$$

$$t^2 \cdot y = \sin(t) + c$$

$y = \frac{\sin(t)}{t^2} + \frac{c}{t^2}$ ← general solution

↳ boundary condition: $y(\pi) = 0$. then:

$\frac{\sin(\pi)}{\pi^2} = -\frac{c}{\pi^2} \implies c = 0.$ thus:

$y = \frac{\sin(t)}{t^2}$ ← particular solution

- numerical approximation

$\frac{dy}{dx} = f(x, y) \implies \Delta y = f(x, y) \Delta x$ if you know how x is changing, you can figure out how y is changing

$x \leftarrow x_0$

$y \leftarrow y_0$

$\Delta x \leftarrow \varepsilon$

$$\left.\begin{array}{l} \Delta y \leftarrow f(x,y)\,\Delta x \\ x \leftarrow x + \Delta x \\ y \leftarrow y + \Delta y \end{array}\right\} \text{repeat}$$

- review

→ partial derivatives and chain rule

$$\begin{tikzcd} & f(x,y) \arrow[dl] \arrow[dr] & \\ x(t) \arrow[dr] & & y(t) \arrow[dl] \\ & t & \end{tikzcd}$$

$$\frac{d}{dt}[f(x,y)] = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \qquad \Big| \cdot dt$$

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy$$

this looks like a DFQ

rewrite:

↳ let $M(x,y) = \frac{\partial}{\partial x}[f(x,y)]$ and $N(x,y) = \frac{\partial}{\partial y}[f(x,y)]$. then:

$$M(x,y)\,dx + N(x,y)\,dy = df$$

↳ if $df = 0 \Rightarrow M(x,y)\,dx + N(x,y)\,dy = 0$. then:

↳ solution:

$$f(x,y) = C$$

→ Q: how do we find this $f(x,y)$?

° separable equations

↳ form: $M(x)\,dx + N(y)\,dy = 0.$ where $\begin{cases} M(x) = \frac{\partial f}{\partial x} \\ \text{and} \\ N(y) = \frac{\partial f}{\partial y} \end{cases}$

$$f(x,y) = \underbrace{\int M(x)\,dx}_{H_1(x)} + \underbrace{\int N(y)\,dy}_{H_2(y)} + C$$

$$f(x,y) = H_1(x) + H_2(y)$$

↳ note: $H_1'(x) = M(x)$ and $H_2'(y) = N(y)$

↳ general solution: $H_1(x) + H_2(y) = C$

① $\frac{dy}{dx} = \frac{x^2}{1-y^2}$

$(1-y^2)\,dy = x^2\,dx$ $\qquad M(x) = -x^2 \qquad N(y) = 1-y^2$

$\qquad\qquad\qquad\qquad\qquad \downarrow \qquad\qquad \downarrow$

$-x^2dx + (1-y^2)\,dy = 0$ $\qquad H_1(x) = -\frac{1}{3}x^3 \qquad H_2(y) = y - \frac{1}{3}y^3$

↳ general solution: $H_1(x) + H_2(y) = C$:

$$-\frac{1}{3}x^3 + y - \frac{1}{3}y^3 = C \quad \Rightarrow \quad -x^3 + 3y - y^3 = C.$$

↳ particular solution: boundary condition: $y(1) = 0$

$$-1 + 0 + 0 = C \Rightarrow C = -1 \quad \Rightarrow \quad -x^3 + 3y - y^3 = -1$$

② $y' + y^2 \sin(x) = 0$ boundary condition: $y(0) = 1$

$\frac{dy}{dx} = -y^2 \sin(x)$ and under $y \neq 0$:

$-\frac{1}{y^2} dy = \sin(x) dx$

$\frac{1}{y^2} dy + \sin(x) dx = 0$ $M(x) = \sin(x)$ $N(y) = \frac{1}{y^2}$.

$$M(x) = \sin(x) \downarrow H_1(x) = -\cos(x) \qquad N(y) = \frac{1}{y^2} \downarrow H_2(y) = -\frac{1}{y}$$

↳ general solution: $H_1(x) = -\cos(x)$ $H_2(y) = -\frac{1}{y}$

$-\cos(x) - \frac{1}{y} = c \implies \cos(x) + \frac{1}{y} = c$

↳ case $y = 0$: $\frac{dy}{dx} = 0$:

$1 dy = 0$ $N(y) = 1 \implies H_2(y) = y$

$y = c.$ thus:

↳ total general solution:
$$\begin{cases} \cos(x) + \frac{1}{y} = c, & \text{for } y \neq 0 \\ y = C, & \text{for } y = 0 \end{cases}$$

?

↳ particular solution: $y(0) = 1$ we'll go over in next class

$\cos(0) + 1 = c \implies c = 2.$ thus: $\cos(x) + \frac{1}{y} = 2.$

③ $\frac{dy}{dx} = \frac{x^2}{y}$

$y\,dy - x^2\,dx = 0.$ $M(x) = -x^2$ $N(y) = y$

$\downarrow$ $\downarrow$

↳ general solution: $H_1(x) = -\frac{1}{3}x^3$ $H_2(y) = \frac{1}{2}y^2$

$-\frac{1}{3}x^3 + \frac{1}{2}y^2 = c \mid \cdot 6$

$-2x^3 + 3y^2 = C$

- exact equations

↳ form: $M(x,y)\,dx + N(x,y)\,dy = 0$

↳ question: does there $\exists$ $f(x,y)$ s.t. $\begin{cases} \frac{\partial}{\partial x}[f(x,y)] = M(x,y) \\ \text{and} \\ \frac{\partial}{\partial y}[f(x,y)] = N(x,y) \end{cases}$

① $\frac{dy}{dx} = \frac{-y}{2y+x}$

$$y\,dx + (2y+x)\,dy = 0$$

Q: does $f(x,y)$ $\exists$ s.t. $\begin{cases} \frac{\partial f}{\partial x} = y & \to f(x,y) = yx + C_1(y) \\ \frac{\partial f}{\partial y} = 2y + x & \to f(x,y) = y^2 + xy + C_2(x) \end{cases}$

↳ $f(x,y) = y^2 + yx + C_3$.

↳ general solution: $df = 0$: $f(x,y) = C$:

$$y^2 + yx = C$$

② $2x + y^2 + 2xy\frac{dy}{dx} = 0$

$$2xy\frac{dy}{dx} = -2x - y^2$$

$$2xy\,dy = -(2x + y^2)\,dx$$

$$2xy\,dy + (2x + y^2)\,dx = 0$$

does $f(x,y)$ $\exists$ s.t. $\begin{cases} \frac{\partial f}{\partial x} = 2x + y^2 \rightarrow f(x,y) = x^2 + xy^2 + C_1(y) \\ \frac{\partial f}{\partial y} = 2xy \rightarrow f(x,y) = y^2x + C_2(x). \end{cases}$

↳ $f(x,y) = x^2 + xy^2 + C_3.$

↳ general solution: $df = 0 \Rightarrow f(x,y) = C$:

$$x^2 + xy^2 = C$$

- linear equations

1) general solution exists w/ an arbitrary constant c

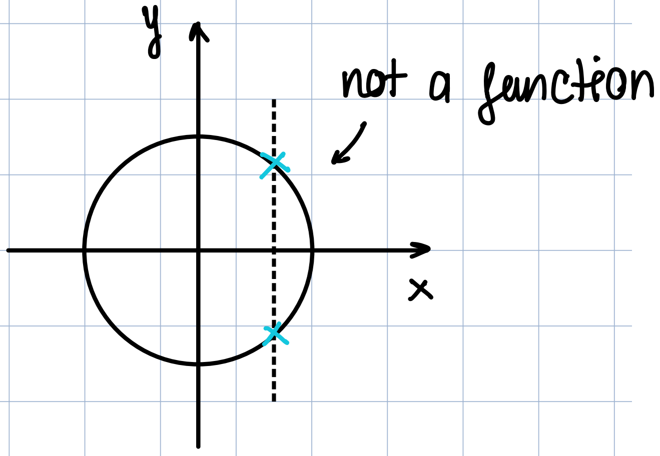
2) there is an explicit expression / formula for the solution

→ note on explicit vs. implicit:

↳ explicit: $y = x^2 + x + \sin(x)$

↳ implicit: $x^2 - y^2 + 2x = C$

$y = \pm\sqrt{x^2 + 2x - C}$



$x^2 + y^2 = c^2 \longrightarrow$ implicit

3) the points of discontinuity / singularity can be identified from the DFQ

- non - linear DFQs

↳ none of the above applies

→ Q: given a DFQ, do we have a solution and is it unique?

- Theorem - Existence and Uniqueness - Linear

↳ for first-order linear ODEs (form: $y' + p(t)y = g(t)$)

↳ if functions $p(t)$ and $g(t)$ are continuous on the interval $\alpha < t < \beta$, containing the initial point $t = t_0$, then:

$\exists$ a unique solution $y = \phi(t)$ for all $t \in (\alpha, \beta)$

which also satisfies the initial condition $y(t_0) = y_0$

→ interpretation:

↳ the given initial value problem has a solution (existence) and only one solution (uniqueness)

↳

① $y' + \frac{2}{t} y = 4t$ → first-order linear ODE

↳ method of integrating factors

$$p(t) = \frac{2}{t} \rightarrow \int p(t)\,dt = 2\int \frac{1}{t}\,dt = 2\ln(t)$$

$$\mu(t) = e^{\int p(t)dt} = e^{2\ln(t)} = \left(e^{\ln(t)}\right)^2 = t^2.$$

↳ multiply both sides by $\mu(t)$:

$$t^2 y' + 2t y = 4t^3$$

$$(t^2 y)' = 4t^3$$

$$t^2 y = \int_{t_0}^{t} 4s^3\,ds + C = s^4 \Big]_{t_0}^{t} + C = t^4 - t_0^4 + C$$

$$y = \frac{1}{t^2}\left(t^4 - t_0^4 + c\right) = t^2 - \frac{t_0^4}{t^2} + \frac{C}{t^2} = \Phi(t) \longleftarrow$$

general solution

↳ initial condition: $y(1) = 2$:

$2 = 1 - 1 + C \Rightarrow C = 2.$ thus:

$$\Phi(t) = t^2 - \frac{1}{t^2} + \frac{2}{t^2}$$

$$\Phi(t) = t^2 + \frac{1}{t^2}.$$

b) use the above Thm to find an interval in which this initial value problem has a unique solution:

$g(t) = 4t$ → continuous everywhere

$p(t) = \frac{2}{t}$ → continuous for $t < 0$ or $t > 0$.

↳ now, given that $t_0 = 1$, the interval that contains this intial point t_0 is the interval $t > 0$. thus:

Theorem 2.4.1 guarantees that this problem will have a unique solution on the interval $0 < t < \infty$.

$\Phi(t) = t^2 + \frac{1}{t^2}$, $t > 0$.

∘ Theorem - Existence and Uniqueness - Non-Linear

↳ for any first-order ODE (form: $y' = f(t, y)$)

↳ if functions f and $\frac{\partial f}{\partial y}$ are continuous in some rectangle $\alpha < t < \beta$ and $\gamma < y < \delta$, conteining the point (t_0, y_0), then:
in some interval $(t_0 - h) < t < (t_0 + h)$ contained in $\alpha < t < \beta$, $\exists$ a unique solution $y = \phi(t)$ of the initial value problem.

↳ observe: this Thm still holds true for linear first-order ODEs

$y' = -p(t) \cdot y + g(t)$. $f(t, y) = -p(t) \cdot y + g(t)$.

$$\frac{\partial f}{\partial y} = -p(t)$$

① $\frac{dy}{dx} = -\frac{x}{y}$ → non-linear, separable $\quad y(0) = \frac{1}{2}$

$y\,dy = -x\,dx$

$M(x) = x \quad N(y) = y$

$\downarrow \quad \downarrow$

$x\,dx + y\,dy = 0$

$H_1(x) = \frac{1}{2}x^2 \quad H_2(y) = \frac{1}{2}y^2$

$\frac{1}{2}x^2 + \frac{1}{2}y^2 = C$

$x^2 + y^2 = C$ ← general solution

↳ boundary condition: $y(0) = \frac{1}{2}$:

$0 + \frac{1}{4} = C \Rightarrow C = \frac{1}{4}$. thus:

$x^2 + y^2 = \frac{1}{4}$ ← particular solution

↳ note: this is implicit ∵ $\quad y = \pm\sqrt{-x^2 + \frac{1}{4}}$

but given that $y(0) = \frac{1}{2}$, we consider the positive part of y: $y = +\sqrt{-x^2 + \frac{1}{4}}$

② $\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$, $y(0) = -1$

$f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}$ $\quad \frac{\partial f}{\partial y} = \frac{3x^2 + 4x + 2}{2} \cdot \left(\frac{1}{y-1}\right)'$

$$= -\frac{3x^2 + 4x + 2}{2(y-1)^2}$$

↳ both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous everywhere besides the line $y = 1$. thus:

we can draw a rectangle around the initial point $(0,-1)$.

we'll solve the DFQ to see the dimensions of the rect.

$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$

$M(x) = 3x^2 + 4x + 2$

$\downarrow$

$H_1(x) = x^3 + 2x^2 + 2x$

$(2y - 2)\,dy - (3x^2 + 4x + 2)\,dx = 0$

$N(y) = 2y - 2$

$\downarrow$

$H_2(y) = y^2 - 2y$

$y^2 - 2y - x^3 - 2x^2 - 2x = C$

$1 + 2 = C \Rightarrow C = 3.$

$y^2 - 2y = x^3 + 2x^2 + 2x + 3$

$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + 4$

$(y-1)^2 = x^3 + 2x^2 + 2x + 4$

$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$ but $\because$ $y(0) = -1$, we choose the negative one

$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$

↳ to find the interval in which this solution is valid, $x^3 + 2x^2 + 2x + 4$ can't be negative:

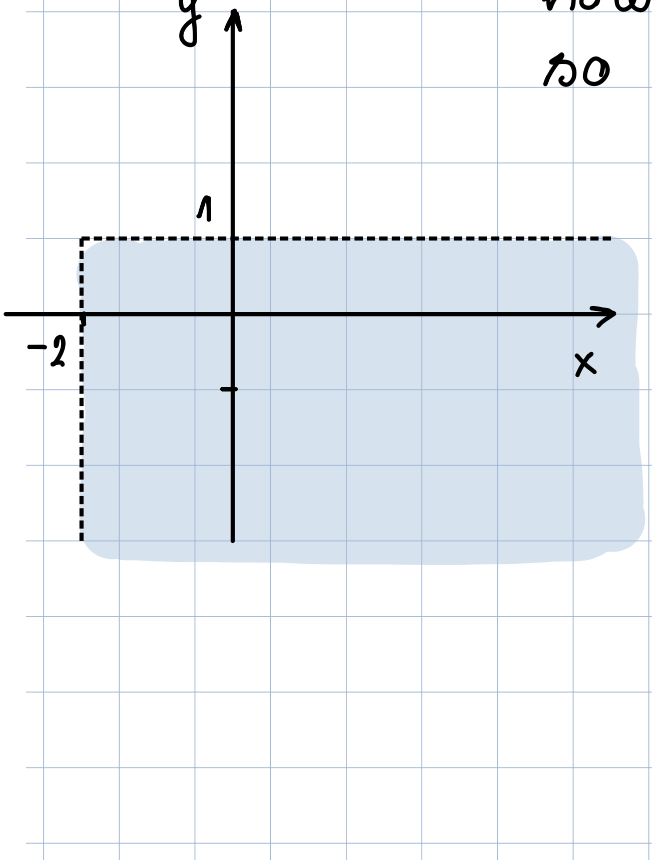
$x^3 + 2x^2 + 2x + 4 = 0$

$x^2(x+2) + 2(x+2) = 0$

$(x^2+2)(x+2) = 0$

$x = -2$. thus, for a non-negative quantity under the radical, $x \geq -2$.

however, $x = -2$ would yield $y = 1$, so we only choose $x > -2$.



- second order linear DFQs

↳ form: $y'' + p(t)y' + q(t)y = g(t)$

or $P(t)y'' + Q(t)y' + R(t)y = G(t)$

- homogeneous second-order DFQs w/ constant coefficients

$ay'' + by' + cy = 0$, where $a, b, c \rightarrow$ const.

→ if we get an exponential solution, $y = y_0 \cdot e^{rt}$, then:

↳ $ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0 \Rightarrow e^{rt}(ar^2 + br + c) = 0$. thus:

↳ characteristic equation: $ar^2 + br + c = 0$

↳ from here, find the roots r_1 and r_2 :

1) case 1: the discriminant $b^2 - 4ac > 0$. then:

↳ we'll be able to find two real, unequal roots $r_1 \neq r_2$.

↳ general solution: $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

and to get C_1 and C_2, plug in the initial conditions

① a) for what values of r is the function e^{rt} a solution for

$ay'' + by' + cy = 0$, where a, b, c are constants

$y = e^{rt}$ $\quad y' = r \cdot e^{rt}$ $\quad y'' = r^2 e^{rt}$

$a \cdot r^2 e^{rt} + b \cdot re^{rt} + ce^{rt} = 0 \quad \Rightarrow \quad e^{rt}(ar^2 + br + c) = 0$

$ar^2 + br + c = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

b) give a general form of solutions for $y'' + y' - 6y = 0$
hint: this includes constant parameters c_1 and c_2.

$r^2 + r - 6 = 0$ $\quad \because a = 1, b = 1, c = -6$

$r^2 + 3r - 2r - 6 = 0$

$(r-2)(r+3) = 0 \quad \Rightarrow \quad r_1 = 2, \quad r_2 = -3.$ thus:

$y(t) = c_1 e^{2t} + c_2 e^{-3t} \leftarrow$ general solution

c) does the above method work for a non-homogeneous equation: $ay'' + by' + cy = g(t)$

↳ nope, because then you can't factor e^{rt}.

② find the solution for the initial value problem:

$y'' - 5y' + 6y = 0$ w/ boundary conditions
$y(0) = 2$ and $y'(0) = 3$

↳ characteristic equation: $r^2 - 5r + 6 = 0$
$(r-2)(r-3) = 0$. $r_1 = 2$ $r_2 = 3$.

↳ general solution: $y(t) = C_1 e^{2t} + C_2 e^{3t}$

↳ particular solution: $y(0) = 2$ and $y'(0) = 3$

$2 = C_1 + C_2$ and $y'(t) = 2C_1 e^{2t} + 3C_2 e^{2t}$, so:
$3 = 2C_1 + 3C_2$

$$\left.\begin{aligned} C_1 + C_2 &= 2 \\ 2C_1 + 3C_2 &= 3 \end{aligned}\right\}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 3 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

$C_1 = 3$
$C_2 = -1$

thus:

$y(t) = 3e^{2t} - e^{3t}$.

③ find the solution for the initial value problem:

$4y'' - 8y' + 3y = 0 \quad , \; y(0) = 2 \; , \quad y'(0) = \frac{1}{2} \quad ; \; y = e^{rt}$

↳ characteristic equation: $4r^2 - 8r + 3 = 0.$

$4r^2 - 6r - 2r + 3 = 0 \rightarrow (2r-1)(2r-3) = 0 \rightarrow \begin{matrix} r_1 = \frac{1}{2} \\ r_2 = \frac{3}{2} \end{matrix}$

↳ general solution: $y(t) = C_1 e^{\frac{1}{2}t} + C_2 e^{\frac{3}{2}t}$

↳ boundary conditions:

$y'(t) = \frac{1}{2} C_1 e^{\frac{1}{2}t} + \frac{3}{2} C_2 e^{\frac{3}{2}t}$; for $y'(0) = \frac{1}{2}$:

$\frac{1}{2} = \frac{1}{2} C_1 + \frac{3}{2} C_2$ and $2 = C_1 + C_2$:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 3 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 2 & -1 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1/2 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & 0 & 5/2 \\ 0 & 1 & -1/2 \end{array}\right] \quad \begin{matrix} C_1 = \frac{5}{2} \\ C_2 = -\frac{1}{2} \end{matrix} .$$

↳ particular solution:

$y(t) = \frac{5}{2} e^{\frac{1}{2}t} - \frac{1}{2} e^{\frac{3}{2}t}$

2) case 2: the discriminant $b^2 - 4ac = 0$. then:

↳ we have 1 real root, r, of the characteristic equation.

↳ general solution: $y(t) = C_1 e^{rt} + C_2 \cdot t \cdot e^{rt}$

and to get C_1 and C_2, plug in the initial conditions

◦ complex numbers

↳ form: $a + bi$, where $i^2 = -1$

→ properties:

1) addition:

↳ $(2+3i) + (5+6i) = 7+9i$

2) multiplication:

↳ $(2+4i)(4+5i) = 8 + 10i + 16i + 20(-1) = -12 + 26i$

→ the equation $ax^2 + bx + c = 0$ always has 2 solutions in complex numbers

① $x^2 + 1 = 0$
$x^2 = -1$ ⤴ $x^2 = i^2$
$x = \pm i$ → $x_1 = i$ and $x_2 = -i$

② $x^2 + x + 2 = 0$ → $a = 1$, $b = 1$, $c = 2$.

$$x = \frac{-1 \pm \sqrt{1 - 4 \cdot 2}}{2} = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm i\sqrt{7}}{2}$$

$$x_1 = \frac{-1 + i\sqrt{7}}{2} \qquad x_2 = \frac{-1 - i\sqrt{7}}{2}$$

③ a) what could be a reasonable solution to

$$\frac{dy}{dt} = iy \quad , \quad y(0) = 1$$

$$\frac{1}{y}\, dy = i\, dt$$

$$\int \frac{1}{y}\, dy = \int i\, dt$$

$$\ln(y) = it + C$$

$$y = e^{it+C}$$

$y(t) = C \cdot e^{it}$ ← general solution

↳ boundary condition: $y(0) = 1$:

$1 = C \implies y(t) = e^{it}$ ← particular solution

b) show that $y(t) = \cos(t) + i \cdot \sin(t)$ is another solution

$$\frac{dy}{dt} = i \cdot y$$

$$-\sin(t) + i \cdot \cos(t) = i \cdot (\cos(t) + i \cdot \sin(t))$$

$$i \cdot \cos(t) - \sin(t) = i \cdot \cos(t) - \sin(t) \quad \blacksquare$$

c) given parts a) and b), give a formula relating the exponential function and cos and sin:

↳ notice how $\frac{dy}{dt} - i \cdot y = 0$ is a linear DFQ with $p(t) = -i$ and $g(t) = 0$. since $p(t)$ and $g(t)$ are continuous $\forall t$, this guarantees a unique solution $\forall t$.

↳ since there has to be only 1 IVP solution (Thm. E & U):

$e^{it} = \cos(t) + i \cdot \sin(t)$ → this is called Euler's formula

d) what about the case when $t = \pi$?

$$e^{i\pi} = -1 \quad \Rightarrow \quad e^{i\pi} + 1 = 0.$$

∘ characteristic equation w/ complex roots:

① $y'' + y' + y = 0$ $\qquad y(0) = 0 \ , \ y'(0) = 1$

↳ characteristic equation: $r^2 + r + 1 = 0$

↳ roots: $r = \frac{-1 \pm \sqrt{1-4}}{2}$

$r_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $r_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$. thus:

$$y_1(t) = e^{r_1 t} = e^{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)t} = e^{-\frac{1}{2}t} \cdot e^{\frac{i\sqrt{3}}{2}t}$$

by Euler's formula $e^{it} = \dots$

$$= e^{-\frac{1}{2}t} \cdot \left(\cos\left(\frac{\sqrt{3}}{2} \cdot t\right) + i \cdot \sin\left(\frac{\sqrt{3}}{2} t\right)\right) \longleftarrow r_1$$

$$y_2(t) = e^{-\frac{1}{2}t} \cdot \left(\cos\left(-\frac{\sqrt{3}}{2} t\right) + i \cdot \sin\left(-\frac{\sqrt{3}}{2} \cdot t\right)\right) \longleftarrow r_2$$

$$= e^{-\frac{1}{2}t} \cdot \left(\cos\left(\frac{\sqrt{3}}{2} t\right) - i \cdot \sin\left(\frac{\sqrt{3}}{2} t\right)\right)$$

↳ general complex solution: $y(t) = C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}$

$$y(t) = C_1 \cdot y_1(t) + C_2 \cdot y_2(t)$$

$$y(t) = C_1 \cdot e^{-\frac{1}{2}t} \cdot \left(\cos\left(\frac{\sqrt{3}}{2} \cdot t\right) + i \cdot \sin\left(\frac{\sqrt{3}}{2} t\right)\right) +$$

$$+ C_2 \cdot e^{-\frac{1}{2}t} \cdot \left(\cos\left(\frac{\sqrt{3}}{2} t\right) - i \cdot \sin\left(\frac{\sqrt{3}}{2} t\right)\right)$$

↳ general, real-valued solution: $C_1, C_2 \in \mathbb{R}$

$$y(t) = C_1 \cdot e^{-\frac{1}{2}t} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \cdot e^{-\frac{1}{2}t} \cdot \sin\left(\frac{\sqrt{3}}{2}t\right)$$

② $y'' + y = 0$. $a = 1$, $b = 0$, $c = 1$. thus:

$r^2 + 1 = 0 \rightarrow r^2 = -1 \rightarrow r_1 = i$, $r_2 = -i$.

↳ general solution: since $\lambda = 0$ and $\mu = 1$:

$$y(t) = c_1 e^0 \cdot \cos(1t) + c_2 e^0 \cdot \sin(1t)$$

$$y(t) = C_1 \cdot \cos(t) + C_2 \cdot \sin(t)$$

3) case 3: the discriminant $b^2 - 4ac < 0$. then:

↳ we have 2 complex roots of the characteristic equation:

$$r_1 = \lambda + \mu i \qquad r_2 = \lambda - \mu i$$

↳ general solution:

$$y(t) = c_1 e^{\lambda t} \cdot \cos(\mu t) + c_2 e^{\lambda t} \cdot \sin(\mu t)$$

and to get c_1 and c_2, plug in the initial conditions

◦ Existence and Uniqueness Theorem

↳ the IVP $y'' + p(t)y' + q(t)y = g(t)$; $y(t_0) = y_0$ $y'(t_0) = y'_0$ has a unique solution $y = \Phi(t)$ on any open time interval I, where: $p(t)$, $q(t)$, and $g(t)$ are continuous ; $t_0 \in I$.

→ interpretation:

↳ if functions p, q, and g are continuous on an open interval I that conteins the point p_0, then:

1) the IVP has a solution
2) the solution is unique
3) the solution Φ is defined throughout the interval I where the coefficients (p, q, and g) are cont. and Φ is at least twice differentiable there

① $y'' + \frac{1}{t-3}y' + \frac{t+3}{t(t-3)}y = 0$. $y(1) = 2$, $y'(t) = 1$.

↳ p, q, and g are continuous for $t \neq 0$ and $t \neq 3$. thus: $t \in (-\infty, 0) \cup (0, 3) \cup (3, \infty)$.

↳ since $t_0 = 1$, $I = (0, 3)$. thus: this IVP has a unique solution on the interval $t \in (0, 3)$.

- the Wronskian

↳ suppose that y_1 and y_2 are two solutions of

$$y'' + p(t)\cdot y' + q(t)\cdot y = 0 \quad w/ \quad y(t_0) = y_0 \ , \ y'(t_0) = y_0'$$

↳ then, finding a specific solution $y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$ is only possible if :

the Wronskian, $W = y_1(t_0)\cdot y_2'(t_0) - y_2(t_0)\cdot y_1'(t_0)$ is $\neq 0$

$$W = \det\left(\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}\right) = y_1(t_0)\cdot y_2'(t_0) - y_2(t_0)\cdot y_1'(t_0).$$

↳ if we can find a t s.t. $W \neq 0$, we have a unique solution.

② $y'' + 5y' + 6y = 0$.

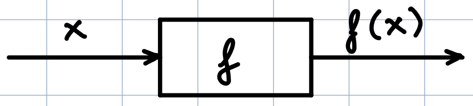
$r^2 + 5r + 6 = 0 \rightarrow (r+2)(r+3) = 0 \rightarrow r_1 = -2, \; r_2 = -3$.

$y_1(t) = e^{-2t}, \quad y_2(t) = e^{-3t}$.

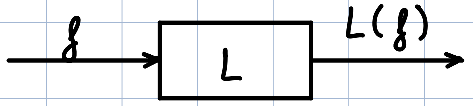
$$W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -3e^{-5t} + 2e^{-5t} = -e^{-5t} \neq 0.$$

↳ since $W \neq 0 \; \forall t$, we'll be able to find a unique solution:

$$y(t) = C_1 e^{-2t} + C_2 e^{-3t} \quad \forall t.$$



a function



an operator

↳ linear function: $f(x_1 + x_2) = f(x_1) + f(x_2)$
$f(cx) = cf(x)$

↳ linear operator: $L(u_1 + u_2) = L(u_1) + L(u_2)$
$L(cu) = cL(u)$

① $Lu = \frac{d^2u}{dt^2} + \cos(t)\cdot\frac{du}{dt} + u$. is L a linear operator?

↳ yes (the 2 conditions apply)

② $Lu = u\cdot\frac{du}{dt}$. is L a linear operator?

1) $L(u_1 + u_2) = (u_1 + u_2)\left(\frac{du_1}{dt} + \frac{du_2}{dt}\right)$

$$= u_1\cdot\frac{du_1}{dt} + u_1\cdot\frac{du_2}{dt} + u_2\cdot\frac{du_1}{dt} + u_2\frac{du_2}{dt}$$

$Lu_1 + Lu_2 = u_1\cdot\frac{du_1}{dt} + u_2\frac{du_2}{dt}$ and $\because L(u_1 + u_2) \neq Lu_1 + Lu_2$:
this L is not linear.

- non-homogeneous linear DFQ s

1) $y'' + p(t)y' + q(t)y = g(t)$, where $g(t) \neq 0$

2) $y'' + p(t)y' + q(t)y = 0$ → homogeneous

↳ let $Lu = u'' + p(t)u' + q(t)u$. then:

1) $Lu = g$ and 2) $Lu = 0$

↳ assume $Lu = g$ has two solutions: Y_1 and Y_2 :

$LY_1 = g$ and $LY_2 = g$

↳ let's look at the difference $Y_2(t) - Y_1(t)$:

$L(Y_2 - Y_1) \overset{\text{linearity}}{=} LY_2 - LY_1 = g - g = 0$. thus:

$Y_2(t) - Y_1(t)$ is a solution to the homogeneous equation $Lu = 0$.

but we also know how to find the general solution to a homogeneous equation: $y(t) = C_1 \cdot y_1(t) + C_2 \cdot y_2(t)$. thus:

$Y_2(t) - Y_1(t) = C_1 \cdot y_1(t) + C_2 \cdot y_2(t)$. therefore:

$Y_2(t) = C_1 \cdot y_1(t) + C_2 \cdot y_2(t) + Y_1(t)$.

- Theorem ~ introduces the method of undetermined coefficients

↳ if Y_1 and Y_2 are two solutions of the nonhomogeneous equation: $Ly = y'' + p(t)\,y' + q(t)\,y = g(t)$, then:

1) their difference, $Y_1 - Y_2$, is a solution to the corresponding homogeneous equation: $Ly = y'' + p(t)\,y' + q(t)\,y = 0$

2) and ∵ we know that y_1 and y_2 are also solutions to the homogeneous equation $y'' + p(t)\,y' + q(t)\,y = 0$

3) the general solution to the non-homogeneous equation $Lu = g$

is: $y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t) + Y(t)$, where $Y(t)$ is a particular solution of the n-h. $Lu = g$

① $y'' + 7y' + 12y = 3 \cdot e^{2t}$ find a general solution to this

↳ find complementary solutions of the homogeneous eq:

$y'' + 7y' + 12y = 0$

$r^2 + 7r + 12 = 0 \rightarrow (r+4)(r+3) = 0 \quad r_1 = -3$ and $r_2 = -4$

$y(t) = C_1 e^{-3t} + C_2 e^{-4t}$ → general solution to the homogeneous eq.

↳ find a particular solution for the OG eq. $y'' + 7y' + 12y = 3 \cdot e^{2t}$

a solution will be some $u(t) = A \cdot e^{2t}$, A → undetermined coefficient

$u'(t) = 2Ae^{2t}$, $u''(t) = 4Ae^{2t}$. plugging this in:

$4A \cdot e^{2t} + 14A \cdot e^{2t} + 12A \cdot e^{2t} = 3e^{2t}$

$e^{2t}(4A + 14A + 12A) = 3e^{2t} \rightarrow 30A = 3 \rightarrow A = \frac{1}{10}$.

thus, a particular solution: $Y(t) = \frac{1}{10} \cdot e^{2t}$

↳ general solution to the non-homogeneous DFQ:

$y(t) = C_1 e^{-3t} + C_2 e^{-4t} + \frac{1}{10} e^{2t}$

→ if $g(t) = \mathbb{P}_n(t)$ → polynomial w/ degree n

↳ finding a particular solution, $Y(t)$, for $ay'' + by' + cy = g$

form: $Y(t) = (A_0 + A_1 t + ... + A_n t^n) \cdot t^s$; $s = 0, 1, 2$

③ $ay'' + by' + cy = 5t^2 + 3t + 2.$

let $u(t) = A_0 + A_1 t + A_2 t^2$. then:

$u'(t) = A_1 + 2A_2 t$ $u''(t) = 2A_2.$ plugging back in:

$$a(2A_2) + b(A_1 + 2 \cdot A_2 t) + c(A_0 + A_1 t + A_2 t^2) = 2 + 3t + 5t^2.$$

$c \cdot A_2 = 5$. $cA_1 + 2bA_2 = 3$. $cA_0 + bA_1 + 2aA_2 = 2$.

↳ you can solve this system of equations (3 unknowns & 3 eq.s)

↳ but we run into a few issues:

if 1) $c = 0$, $b \neq 0$ we need an extra factor of t on the left:
$Y(t) = (A_0 + A_1 t + ... + A_n t^n) \cdot t^1$

2) $c = 0$, $b = 0$ we need an extra factor of t^2 on the left:
$Y(t) = (A_0 + A_1 t + ... + A_n t^n) \cdot t^2$

↳ if $g(t) = \mathbb{P}_n(t) \cdot e^{\alpha t}$, use the particular $Y(t)$:

$$Y(t) = t^s \left(A_0 + A_1 t + \ldots + A_n t^n \right) \cdot e^{\alpha t}$$

↳ if $g(t) = \mathbb{P}_n(t) \cdot e^{\alpha t} \cdot \cos(\beta t)$, use the particular:

$$Y(t) = t^s \left(A_0 + A_1 t + \ldots + A_n t^n \right) \cdot e^{\alpha t} \cdot \cos(\beta t) + $$
$$+ t^s \left(B_0 + B_1 t + \ldots + B_n t^n \right) \cdot e^{\alpha t} \cdot \sin(\beta t)$$

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

↳ we have complementary solutions for the homogeneous eq:

$$y_c(t) = C_1 \cdot y_1(t) + C_2 \cdot y_2(t)$$

↳ look for the particular solutions of the form:

$$Y(t) = u_1(t) \cdot y_1(t) + u_2(t) \cdot y_2(t)$$

with the condition $u_1' \cdot y_1 + u_2' \cdot y_2 = 0$.

↳ let $Y = u_1 \cdot y_1 + u_2 \cdot y_2$; then:

$$Y' = u_1' \cdot y_1 + u_1 \cdot y_1' + u_2' y_2 + u_2 \cdot y_2'$$ and ∵ of the condition:

$$= u_1 \cdot y_1' + u_2 \cdot y_2'$$

$$Y'' = u_1' \cdot y_1' + u_1 \cdot y_1'' + u_2' \cdot y_2' + u_2 \cdot y_2'' \;;$$

↳ plugging these into: $Y'' + pY' + qY = g$:

$$(u_1' \cdot y_1' + u_1 \cdot y_1'' + u_2' \cdot y_2' + u_2 \cdot y_2'') + \ldots$$
$$\ldots + p(u_1 \cdot y_1' + u_2 \cdot y_2') + q(u_1 \cdot y_1 + u_2 \cdot y_2) = g.$$

$$u_1 \underbrace{(y_1'' + p \cdot y_1' + q \cdot y_1)}_{=0} + u_2 \underbrace{(y_2'' + p \cdot y_2' + q \cdot y_2)}_{=0} +$$
$$+ u_1' \cdot y_1' + u_2' \cdot y_2' = g.$$

↳ now, recall that y_1 and y_2 solve the homogeneous eq. thus:

$$u_1' \cdot y_1' + u_2' \cdot y_2' = g \ .$$

↳ we're left w/ a system of equations:

$$\left.\begin{aligned} u_1' \cdot y_1' + u_2' \cdot y_2' &= g \\ u_1' \cdot y_1 + u_2' \cdot y_2 &= 0 \ . \end{aligned}\right\}$$

variables: u_1' and u_2'

↳ the condition

⋮ after solving for u_1' and u_2', we obtain:

$$u_1' = \frac{-y_2 \cdot g}{w(y_1, y_2)} \qquad u_2' = \frac{y_1 \cdot g}{w(y_1, y_2)} \ .$$ thus:

$$u_1 = \int \frac{-y_2 \cdot g}{w(y_1, y_2)} \qquad u_2 = \int \frac{y_1 \cdot g}{w(y_1, y_2)}$$

↳ particular solution:

$$Y = -y_1 \cdot \int \frac{y_2 \cdot g}{w(y_1, y_2)} + y_2 \cdot \int \frac{y_1 \cdot g}{w(y_1, y_2)}$$

↳ general solution: $y(t) = y_c(t) + Y(t)$:

$$y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t) - y_1 \cdot \int \frac{y_2 \cdot g}{w(y_1, y_2)} + y_2 \cdot \int \frac{y_1 \cdot g}{w(y_1, y_2)}$$

① $y'' + 4y = 3 \cdot \csc(t)$

↳ finding complementary solutions for the homogeneous eq:

$y'' + 4y = 0 \qquad r^2 + 4 = 0 \quad \rightarrow \quad r_1 = +2i \, , \; r_2 = -2i$

$$y_c(t) = e^0 [C_1 \cdot \cos(2t) + C_2 \cdot \sin(2t)] = C_1 \cdot \underbrace{\cos(2t)}_{y_1} + C_2 \cdot \underbrace{\sin(2t)}_{y_2}$$

$y_1 = \cos(2t) \, , \quad y_2 = \sin(2t)$

variation of parameters: replace C_1 and C_2 w/ $u_1(t)$ and $u_2(t)$

↳ condition: $u_1' y_1 + u_2' y_2 = 0 \quad \rightarrow \quad u_1' \cos(2t) + u_2' \sin(2t) = 0$

↳ our particular solution will be: $y_p(t) = u_1 \cdot y_1 + u_2 \cdot y_2$.

$y_p(t) = u_1 \cdot \cos(2t) + u_2 \cdot \sin(2t)$

$y_p(t)' = u_1' \cos(2t) - 2u_1 \cdot \sin(2t) + u_2' \sin(2t) + 2u_2 \cdot \cos(2t)$.

$\quad = 2u_2 \cdot \cos(2t) - 2u_1 \cdot \sin(2t)$

$y_p(t)'' = 2u_2' \cdot \cos(2t) - 4u_2 \cdot \sin(2t) - 2u_1' \cdot \sin(2t) - 4u_1 \cdot \cos(2t)$

$\quad = 2(u_2' \cos(2t) - u_1' \sin(2t)) - 4(u_2 \cdot \sin(2t) + u_1 \cdot \cos(2t))$

plugging $y_p(t)''$ and $y_p(t)$ into the original DFQ:

$$2(u_2'\cos(2t) - u_1'\sin(2t)) - 4(u_2 \cdot \sin(2t) + u_1 \cdot \cos(2t)) + \dots$$

$$\dots + 4(u_2 \cdot \sin(2t) + u_1 \cdot \cos(2t)) = 3\csc(t)$$

$2u_2'\cos(2t) - 2u_1'\sin(2t) = 3\csc(t)$ solve for u_1', u_2':

↳ $u_1' = \frac{W_1}{W}$ $u_2' = \frac{W_2}{W}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2\cos^2(2t) + 2\sin^2(2t) = 2(\cos^2(2t) + \sin^2(2t)) = 2.$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \begin{vmatrix} \cos(2t) & 0 \\ -2\sin(2t) & 3\csc(t) \end{vmatrix} = 3\cos(2t) \cdot \csc(t) = 3\cos(2t)/\sin(t)$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin(2t) \\ 3\csc(t) & 2\cos(2t) \end{vmatrix} = -3\sin(2t) \cdot \csc(t) = -3\sin(2t)/\sin(t) = -6\sin(t) \cdot \cos(t)/\sin(t) = -6\cos(t).$$

↳ we have:

$y_c(t) = C_1 \cdot \cos(2t) + C_2 \cdot \sin(2t)$; $y_1 = \cos(2t)$, $y_2 = \sin(2t)$

$W = 2$, $W_1 = -6\cos(t)$, $W_2 = 3\cos(2t)/\sin(t)$

↳ we want: $y_p(t) = u_1 \cdot y_1 + u_2 \cdot y_2$

$$u_1' = \frac{w_1}{w} = \frac{-6\cos(t)}{2} = -3\cos(t)$$

$$u_2' = \frac{w_2}{w} = \frac{3\cos(2t)/\sin(t)}{2} = \frac{3}{2}\,\frac{\cos(2t)}{\sin(t)}$$

$$u_1 = \int -3\cos(t)\,dt = -3\int \cos(t)\,dt = -3\sin(t) + C_1$$

$$u_2 = \frac{3}{2}\int \frac{\cos(2t)}{\sin(t)}\,dt = 3\cos(t) - \frac{3}{2}\ln(|\csc(t) + \cot(t)|) + C_2$$

↳ plugging u_1 and u_2 into $y_p(t)$:

$$y_p(t) = -3\sin(t)\cdot\cos(2t) + \left(3\cos(t) - \frac{3}{2}\ln(|\csc(t) + \cot(t)|)\right)\cdot\sin(2t)$$
$$+ C_1\cdot\cos(2t) + C_2\cdot\sin(2t).$$

- linear algebra review

→ matrix multiplication

1) first way: $AB = [\,A\vec{b}_1 \quad A\vec{b}_2 \quad \dots \quad A\vec{b}_n\,]$

$$\underset{2\times3}{\begin{bmatrix} 2 & -1 & -2 \\ -4 & 2 & -3 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 7 & -3 \end{bmatrix}} = \underset{2\times2}{\begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 \end{bmatrix}}$$

$$A\vec{b}_1 = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -4 \end{bmatrix} - 1\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 7\begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -14 \\ -21 \end{bmatrix} = \overset{\overset{A\vec{b}_1}{\downarrow}}{\begin{bmatrix} -9 \\ -31 \end{bmatrix}}$$

$$A\vec{b}_2 = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = -1\begin{bmatrix} 2 \\ -4 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \overset{\overset{A\vec{b}_2}{\downarrow}}{\begin{bmatrix} 2 \\ 17 \end{bmatrix}}$$

$$AB = \begin{bmatrix} -9 & 2 \\ -31 & 17 \end{bmatrix}$$

2) second way: (i,j) entry in AB is $R_iA \cdot C_jB$; dot-product

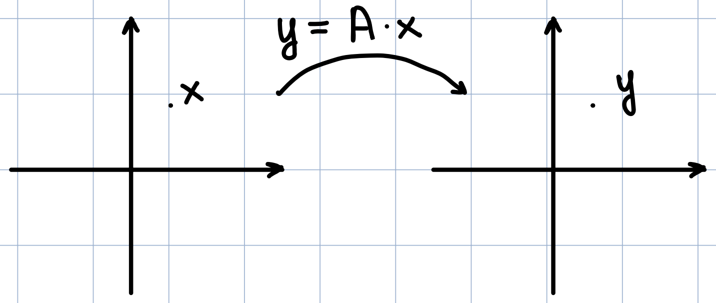
$$\underset{2\times3}{\begin{bmatrix} 2 & -1 & -2 \\ -4 & 2 & -3 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 7 & -3 \end{bmatrix}} = \begin{bmatrix} 2\cdot2+1\cdot1-2\cdot7 & -2\cdot1-1\cdot2+2\cdot3 \\ -4\cdot2-2\cdot1-7\cdot3 & 1\cdot4+2\cdot2+3\cdot3 \end{bmatrix} = \underset{2\times2}{\begin{bmatrix} -9 & 2 \\ -31 & 17 \end{bmatrix}}$$

3) third way: a sum of n rank 1 mxn matrices : CA × RB

$$\underset{2\times3}{\begin{bmatrix}2&-1&-2\\-4&2&-3\end{bmatrix}}\underset{3\times2}{\begin{bmatrix}2&-1\\-1&2\\7&-3\end{bmatrix}} = \begin{bmatrix}2\\-4\end{bmatrix}\begin{bmatrix}2&-1\end{bmatrix}+\begin{bmatrix}-1\\2\end{bmatrix}\begin{bmatrix}-1&2\end{bmatrix}+\begin{bmatrix}-2\\-3\end{bmatrix}\begin{bmatrix}7&-3\end{bmatrix}=$$

$$=\underset{2\times2}{\begin{bmatrix}4&-2\\-8&4\end{bmatrix}}+\underset{2\times2}{\begin{bmatrix}1&-2\\-2&4\end{bmatrix}}+\underset{2\times2}{\begin{bmatrix}-14&6\\-21&9\end{bmatrix}}=\underset{2\times2}{\begin{bmatrix}-9&2\\-31&17\end{bmatrix}}$$

→ linear transformations



$$\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

- determinant

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \det(A) = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

→ geometric interpretation of det:

↪ the $|\det(A)|$ can be thought of as the change of the area of the "unit square" after we apply the lin. transformation A.
↳ basis vectors $\langle 1, 0 \rangle$, $\langle 0, 1 \rangle$ in $\mathbb{R}^2$.

↪ this is why $\det(A) = 0$ means we're losing a dimension (e.g. from $\mathbb{R}^2$, everything is squeezed onto a line → $\mathbb{R}^1$)

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = a_{11} \cdot \det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{12} \cdot \det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right) + \dots$$

$$\dots + a_{13} \cdot \det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) =$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

- eigenvectors, eigenvalues

↳ when $A \cdot \vec{x} = 0$, for $\vec{x} \neq \vec{0}$, this means that $\vec{x}$ gets mapped to the $\vec{0}$-vector. this means that the unit disc is collapsed to a line segment (as a result of a projection along $\vec{x}$)

↳ this $\vec{x} \in Nul(A)$.

↳ for example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

notice that $\det(A) = 0$.

①.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 2 \\ 2 & 1 & 4 \end{bmatrix}, \vec{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad A\vec{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 2 \\ 2 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

notice that $\det(A) = 0(3 \cdot 4 - 2 \cdot 1) - 1(1 \cdot 4 - 2 \cdot 2) + 0 = 0$.

② find vectors $\vec{x}$ and numbers λ s.t. $A\vec{x} = \lambda\vec{x}$:

λ = eigenvalue of A
$\vec{x}$ = eigenvector of A corresponding to λ.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \rightarrow \lambda_1 = 2. \quad \vec{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \quad \lambda_2 = 3. \quad \vec{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\because$ $A\vec{x_1} = \lambda_1\vec{x_1}$ and $A\vec{x_2} = \lambda_2\vec{x_2}$.

↳ this is $\because$ A is diagonal, thus: $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $\vec{x_1} = \vec{e_1}$, $\vec{x_2} = \vec{e_2}$

③ find vectors $\vec{x}$ and numbers λ s.t. $A\vec{x} = \lambda\vec{x}$; $A \neq D$:

$$A\vec{x} = \lambda \cdot I\vec{x}$$

$$A\vec{x} - \lambda \cdot I\vec{x} = 0$$

$(A - \lambda \cdot I\vec{x})\vec{x} = 0$ and $\because$ $\vec{x} \neq 0$:

$\det(A - \lambda \cdot I\vec{x}) = 0$. → use this formula to find λ

$\vec{x}$ will be in the $Nul(A - \lambda \cdot I_n)$

$$A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \rightarrow \det(A) = (3-\lambda)(-2-\lambda) + 4 = \lambda^2 - \lambda - 2$$

$$\lambda^2 - \lambda - 2 = 0 \rightarrow (\lambda - 2)(\lambda + 1) = 0 \text{ . thus:}$$

$$\lambda_1 = 2 \qquad \lambda_2 = -1$$

↳ finding eigenvectors:

1) for $\lambda_1 = 2$: $\text{Nul}(A - 2 \cdot I_n)$:

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 4 & -4 & 0 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so: } \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2) for $\lambda_2 = -1$: $\text{Nul}(A + 1 \cdot I_n)$:

$$\left[\begin{array}{cc|c} 4 & -1 & 0 \\ 4 & -1 & 0 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so: } \vec{x}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

→ one variable calculus

↳ when we zoom in on the graph of a smooth function, we see a line.

→ two variable calculus

↳ when we zoom in on the graph of the multivariable function $f: \mathbb{R}^2 \to \mathbb{R}^2$, we see a linear transformation which approximates ?

$$y_1 = f_1(x_1, x_2) = a_{11} x_1 + a_{12} x_2 + b_1$$
$$y_2 = f_2(x_1, x_2) = a_{21} x_1 + a_{22} x_2 + b_2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- systems of DFQs :

↳ for one variable :

$$y' = ay \quad ; \quad \text{solution: } \quad y(t) = C \cdot e^{at}$$

↳ for two variables:

$y' = A \cdot y$, where $A \rightarrow$ matrix ; $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

↳ for a system:

$$y_1' = a_{11} y_1 + a_{12} y_2$$
$$y_2' = a_{21} y_1 + a_{22} y_2$$

→ goal: come up with a strategy (linear change of variables) to turn any system $\vec{y}' = A \cdot \vec{y}$ into a diagonal form :

$$\vec{y}' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \cdot \vec{y}$$

° first-order linear systems

↳ form: $x' = P(t) \cdot x + g(t)$ if $g(t) = 0$ → homogeneous

$$\underset{\in \mathbb{R}^n}{x'} = \underset{\in \mathbb{R}^{n \times n}}{P(t)} \cdot x + \underset{\in \mathbb{R}^n}{g(t)}$$

$$x_1' = P_{11}(t)\, x_1 + \dots + P_{1n}(t)\, x_n + g_1(t)$$
$$x_2' = P_{21}(t)\, x_1 + \dots + P_{2n}(t)\, x_n + g_2(t)$$
$$\vdots$$
$$x_n' = P_{n1}(t)\, x_1 + \dots + P_{nn}(t)\, x_n + g_n(t)$$

↳ the system has n linearly independent solutions (a vector in $\mathbb{R}^n$)

° first-order, homogeneous linear systems w/ const. coefficients

↳ form: $x' = A x$, where A is a constant matrix

① $\vec{x}' = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \vec{x} \rightarrow \begin{matrix} x_1' = 2x_1 & \rightarrow & x_1 = C_1 e^{2t} \\ x_2' = -3x_2 & \rightarrow & x_2 = C_2 \cdot e^{-3t} \end{matrix}$

↳ checking x_1:

$$\frac{d}{dt}\begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 0 \end{bmatrix} \checkmark$$

↳ checking x_2:

$$\frac{d}{dt}\begin{bmatrix} 0 \\ e^{3t} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix} = \begin{bmatrix} 0 \\ -3e^{-3t} \end{bmatrix} \checkmark$$

↳ thus, the two vector solutions:

$$\vec{x}^1(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{x}^2(t) = e^{-3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

↳ the Wronskian:

$$w[\vec{x}^1, \vec{x}^2](t) = \begin{vmatrix} \underset{\underset{\vec{x}_1}{\uparrow}}{\begin{matrix} e^{2t} \\ 0 \end{matrix}} & \underset{\underset{\vec{x}_2}{\uparrow}}{\begin{matrix} 0 \\ e^{-3t} \end{matrix}} \end{vmatrix} = e^{-t}.$$

↳ since $w[\vec{x}^1, \vec{x}^2](t) \neq 0 \rightarrow \vec{x}^1(t)$ and $\vec{x}^2(t)$ form a fundamental set of solutions.

↳ general solution: $\vec{x}(t) = C_1 \cdot \vec{x}^1(t) + C_2 \cdot \vec{x}^2(t)$

$$\vec{x}(t) = C_1 \cdot e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \cdot e^{-3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

② $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$

↳ assume $\vec{x}(t) = e^{\lambda t} \cdot \xi$ is a solution, where $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$

$$\vec{x}' = \lambda e^{\lambda t} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = e^{\lambda t} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

↳ substituting this:

$$e^{\lambda t} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = e^{\lambda t} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$. thus:

$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$ the only way to have a non-zero $\vec{\xi}$ is if $\det(A - \lambda \cdot I_n) = 0$:

$(1-\lambda)^4 - 4 = 0 \rightarrow \quad \lambda_1 = 3$ and $\lambda_2 = -1$.

1) $\mathcal{E}_3 = Nul(A - 3 \cdot I_n)$:

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \vec{x} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$ so: $\mathcal{E}_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

2) $\mathcal{E}_{-1} = Nul(A + 1 \cdot I_n)$:

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \vec{x} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$ so: $\mathcal{E}_{-1} = \text{span}\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

↳ thus: $\xi^1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\xi^2 = \begin{bmatrix}-1\\2\end{bmatrix}$, so:

$$\vec{x}^1(t) = e^{3t}\begin{bmatrix}1\\2\end{bmatrix} \quad \text{and} \quad \vec{x}^2(t) = e^{-t}\begin{bmatrix}-1\\2\end{bmatrix}$$

↳ the Wronskian:

$$W = \underset{\vec{x}_1 \qquad \vec{x}_2}{\begin{vmatrix} e^{3t} & -e^{-t} \\ 2e^{3t} & 2e^{-t} \end{vmatrix}} = 2e^{2t} + 2e^{2t} = 4e^{2t} \neq 0.$$

↳ since $W[\vec{x}^1, \vec{x}^2](t) \neq 0 \rightarrow \vec{x}^1(t)$ and $\vec{x}^2(t)$ form a fundamental set of solutions.

↳ general solution: $\vec{x}(t) = c_1 \cdot \vec{x}^1(t) + c_2 \cdot \vec{x}^2(t)$

$$\vec{x}(t) = c_1 \cdot e^{3t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 \cdot e^{-t}\begin{bmatrix}-1\\2\end{bmatrix}$$

① special case

$$x' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} x \qquad \det(A - \lambda \cdot \mathcal{I}_n) = (-\lambda)^2 + 4 = \lambda^2 + 4 \ ; \text{ thus:}$$

$$\lambda^2 + 4 = 0 \rightarrow \lambda_1 = 2i \ , \ \lambda_2 = -2i \ .$$

1) $\mathcal{E}_{2i} = \text{Nul}\,(A - 2i \cdot \mathcal{I}_n)$:

$$\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \mapsto \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \mapsto \begin{bmatrix} 1 & \frac{1}{i} \\ 1 & -i \end{bmatrix} \mapsto \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} \mapsto \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \quad \vec{x} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

2) $\mathcal{E}_{-2i} = \text{Nul}\,(A + 2i \cdot \mathcal{I}_n)$:

$$\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix} \mapsto \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \mapsto \begin{bmatrix} 1 & \frac{-1}{i} \\ 1 & i \end{bmatrix} \mapsto \begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix} \mapsto \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \quad \vec{x} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

↳ general solution: $\vec{x}(t) = c_1 \cdot \vec{x}^1(t) + c_2 \cdot \vec{x}^2(t)$, where:

$$\vec{x}^1(t) = e^{\lambda_1 \cdot t} \cdot \begin{bmatrix} \text{eig.} \\ \text{vec.} \end{bmatrix} \ , \quad \vec{x}_2(t) = e^{\lambda_2 \cdot t} \cdot \begin{bmatrix} \text{eig.} \\ \text{vec.} \end{bmatrix}$$

$$\vec{x}(t) = c_1 \cdot e^{2it} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 \cdot e^{-2it} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$W = \begin{vmatrix} ie^{2it} & -ie^{-2it} \\ e^{2it} & e^{-2it} \end{vmatrix} = ie^0 + ie^0 = 2i \neq 0$$

↳ thus, $\vec{x}^1(t)$ and $\vec{x}^2(t)$ → fundamental s.

- fundamental matrix of a system

↳ for a system $x' = Ax$, the fundamental matrix $\Phi(t)$:

$$\Phi(t) = [\, \vec{x}^1(t) \quad \vec{x}^2(t) \,]$$ → $\vec{x}^1$, $\vec{x}^2$ are column vectors

→ now, we're looking for a solution in terms of matrices:

↳ $\Phi(t)$ represents a matrix solution to the system

↳ it's because $\Phi(t)$ is a "linear combination" of $\vec{x}^1$ and $\vec{x}^2$, which are linearly independent → $\Phi(t)$ is invertible as well.

↳ in the space of matrices, you now only need to specify one boundary condition to fully describe the IVP of 1st-order sys.

↳ for example:

for $x' = ax$, solution: $x = e^{at}$

for $x' = Ax$, solution: $x = e^{At}$

- Taylor expansion:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

① $x' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x$ $\det(A - \lambda \cdot I_n) = (-\lambda)^2 + 1 = \lambda^2 + 1$. thus:

$\lambda^2 + 1 = 0 \rightarrow \lambda_1 = i \,, \; \lambda_2 = -i$

1) $\mathcal{E}_i = Nul\,(A - i \cdot I_n)$:

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \mapsto \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \mapsto \begin{bmatrix} 1 & \frac{1}{i} \\ 1 & -i \end{bmatrix} \mapsto \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} \mapsto \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \quad \vec{x} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

2) $\mathcal{E}_{-i} = Nul\,(A + i \cdot I_n)$:

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \mapsto \begin{bmatrix} 1 & \frac{-1}{i} \\ 1 & i \end{bmatrix} \mapsto \begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix} \mapsto \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \quad \vec{x} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

↳ general solution: $\vec{x}(t) = c_1 \cdot \vec{x}^1(t) + c_2 \cdot \vec{x}^2(t)$, where:

$$\vec{x}^1(t) = e^{\lambda_1 \cdot t} \cdot \begin{bmatrix} \text{eig.} \\ \text{vec.} \end{bmatrix} \quad , \quad \vec{x}_2(t) = e^{\lambda_2 \cdot t} \cdot \begin{bmatrix} \text{eig.} \\ \text{vec.} \end{bmatrix}$$

$$\vec{x}(t) = c_1 \cdot e^{it} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 \cdot e^{-it} \begin{bmatrix} -i \\ 1 \end{bmatrix} \rightarrow \Phi(t) = \begin{bmatrix} ie^{it} & -ie^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

↳ $\Phi(t)$ is a solution to the matrix equation $x' = Ax$.

$$w = \begin{vmatrix} ie^{it} & -ie^{-it} \\ e^{it} & e^{-it} \end{vmatrix} = ie^{0} + ie^{0} = 2i \neq 0$$

↳ thus, $\Phi(t) \rightarrow$ fundamental solution

↳ verify that $\Phi(t)$ is a solution to this DFQ:

$$\Phi(t) = \begin{bmatrix} ie^{it} & -ie^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \qquad \Phi'(t) = \begin{bmatrix} -e^{it} & -e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix}$$

↳ plugging into $\Phi'(t) = A \cdot \Phi(t)$:

$$\begin{bmatrix} -e^{it} & -e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} ie^{it} & -ie^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

$$\begin{bmatrix} -e^{it} & -e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix} = \begin{bmatrix} -e^{it} & -e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix} \checkmark$$

↳ initial condition: $\Phi(0) = I_n$: thus, we're looking for $\tilde{\vec{x}}^1(t)$ and $\tilde{\vec{x}}^2(t)$ s.t. $\tilde{\vec{x}}^1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\tilde{\vec{x}}^2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\tilde{\vec{x}}^1(t) = C_{11} \cdot \vec{x}^1(t) + C_{12}\, \vec{x}^2(t)$$

$$\tilde{\vec{x}}^2(t) = C_{21} \cdot \vec{x}^1(t) + C_{22}\, \vec{x}^2(t)$$

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

↳ note: since $\vec{x}^1(t) = e^{it} \begin{bmatrix} i \\ 1 \end{bmatrix}$, $\vec{x}^1(0) = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and similarly for $\vec{x}^2(0)$.

1) $\tilde{\vec{x}}^1(t) = c_{11} \cdot \vec{x}^1(t) + c_{12}\,\vec{x}^2(t)$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_{11} \cdot \begin{bmatrix} i \\ 1 \end{bmatrix} + c_{12} \cdot \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} i & -i & 1 \\ 1 & 1 & 0 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{i} \\ 1 & 1 & 0 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & -1 & -i \\ 1 & 1 & 0 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & -1 & -i \\ 0 & 2 & i \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & 0 & -\frac{i}{2} \\ 0 & 1 & \frac{i}{2} \end{array}\right]$$

$c_{11} = -\frac{i}{2}$, $c_{12} = \frac{i}{2}$. thus: $\tilde{\vec{x}}^1(t) = -\frac{i}{2} \cdot \vec{x}^1(t) + \frac{i}{2}\,\vec{x}^2(t)$:

$$\tilde{\vec{x}}^1(t) = -\frac{i}{2}\, e^{it} \begin{bmatrix} i \\ 1 \end{bmatrix} + \frac{i}{2}\, e^{-it} \begin{bmatrix} -i \\ 1 \end{bmatrix} = e^{it} \begin{bmatrix} \frac{1}{2} \\ -\frac{i}{2} \end{bmatrix} + e^{-it} \begin{bmatrix} \frac{1}{2} \\ \frac{i}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{it} + e^{-it} \\ -ie^{it} + ie^{-it} \end{bmatrix}$$

↳ simplifying $\tilde{\vec{x}}^1(t)$:

a) $\frac{1}{2}(e^{it} + e^{-it}) = \frac{1}{2}(\cos(t) + i \cdot \sin(t) + \cos(t) - i \cdot \sin(t)) = \cos(t)$

b) $\frac{1}{2}(-ie^{it} + ie^{-it}) = \frac{i}{2}(-e^{it} + e^{-it}) = \frac{i}{2}(-\cos(t) - i\sin(t) + \cos(t) - i\sin(t))$

$= \frac{i}{2}(-2i \cdot \sin(t)) = -i^2 \sin(t) = \sin(t)$. thus:

$$\tilde{\vec{x}}^1(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

2) $\tilde{\vec{x}}^2(t) = C_{21} \cdot \vec{x}^1(t) + C_{22} \, \vec{x}^2(t)$:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = C_{21} \cdot \begin{bmatrix} i \\ 1 \end{bmatrix} + C_{22} \cdot \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} i & -i & 0 \\ 1 & 1 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{array}\right] \qquad \begin{array}{l} C_{21} = \frac{1}{2} \\ C_{22} = \frac{1}{2} \end{array}$$

thus: $\tilde{\vec{x}}^2(t) = \frac{1}{2} \, \vec{x}^1(t) + \frac{1}{2} \, \vec{x}^2(t)$:

$$\tilde{\vec{x}}^2(t) = \frac{1}{2} \, e^{it} \begin{bmatrix} i \\ 1 \end{bmatrix} + \frac{1}{2} \cdot e^{-it} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} ie^{it} - ie^{-it} \\ e^{it} + e^{-it} \end{bmatrix}$$

↳ simplifying $\tilde{\vec{x}}^2(t)$:

a) $\frac{1}{2}(ie^{it} - ie^{-it}) = \frac{i}{2}(e^{it} - e^{-it}) = \frac{i}{2}(\cos(t) + i \cdot \sin(t) - \ldots$
$\ldots - (\cos(t) - i \cdot \sin(t)) = \frac{i}{2}(2i \cdot \sin(t)) = -\sin(t)$

b) $\frac{1}{2}(e^{it} + e^{-it}) = \frac{1}{2}(\cos(t) + i \cdot \sin(t) + \cos(t) - i \cdot \sin(t)) = \cos(t)$

$$\tilde{\vec{x}}^2(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

↳ therefore:

$$\Phi(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

particular solution to the IVP w/ cond. $\Phi(0) = I_n$.

b) identify odd / even patterns in the powers of A

$$\begin{array}{cccc} A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & A^1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & A^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \mathcal{I}_n & A & -\mathcal{I}_n & -A \end{array}$$

$$\begin{array}{cccc} A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & A^5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \dots & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ \downarrow & \downarrow & & \\ \mathcal{I}_n & A & & \end{array}$$

↳ we can see a pattern which repeats every 4 times:
$A, -\mathcal{I}_n, -A, \mathcal{I}_n$

c) calculate $e^{At} = \mathcal{I}_n + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \frac{(At)^4}{4!} + \dots$

$$\mathcal{I}_n = A^0 = A^4 = A^8 = A^{12} = \dots \; ; \quad A = A^1 = A^5 = A^9 = A^{13} = \dots$$

$$-\mathcal{I}_n = A^2 = A^6 = A^{10} = A^{14} = \dots \; ; \quad -A = A^3 = A^7 = A^{11} = A^{15} = \dots$$

$$e^{At} = \mathcal{I}_n + \frac{(At)^2}{2!} + \frac{(At)^4}{4!} + \dots + At + \frac{(At)^3}{3!} + \frac{(At)^5}{5!} + \dots =$$

$$= \mathcal{I}_n \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots\right) + A\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\right) =$$

$$= \cos(t) \cdot \mathcal{I}_n + \sin(t) \cdot A$$

d) compare $\Phi(t)$ and e^{At}

$e^{At} = \cos(t) \cdot I_n + \sin(t) \cdot A = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$. thus:

$e^{At} = \Phi(t)$

- repeated eigenvalues

$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ $\det(A - \lambda \cdot I_n) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$.
thus: $\lambda = 2$ → only one eigenvalue

↪ we use eigenvalues and eigenvectors to write the general solution:

$$\vec{x}(t) = c_1 \cdot e^{\lambda_1 t} \cdot \xi^1 + c_2 \cdot e^{\lambda_2 t} \cdot \xi^2$$

↪ recall: each eigenvalue guarantees at least 1 eigenvector

↪ when working in $\mathbb{C}$, we'll always be able to find n lin. ind. eigenvectors ($\forall$ matrices A)

↪ bad news: if λ is a repeated eigenvalue w/ algebraic multiplicity 2, it's possible that we have only 1 eigenvector (geometric multiplicity < algebraic)

① $x' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} x \rightarrow \lambda_1 = \lambda_2 = 2$

1) $\mathcal{E}_2 = \text{Nul}(A - 2 \cdot I_n)$:

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \mathcal{E}_2 = \text{span}\left\{ \overset{\overset{\xi}{\downarrow}}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \right\}$$

$\vec{x}^1(t) = e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ but what about $\vec{x}^2(t)$?

↳ possible form for $\vec{x}^{2}(t)$: a combination of two vectors:

$$\vec{x}^{2}(t) = t \cdot e^{2t} \xi + e^{2t} \eta$$

↳ plugging this into the OG eq $x' = Ax$:

$$x' = e^{2t} \cdot \xi + 2te^{2t} \cdot \xi + 2e^{2t} \eta = e^{2t}(\xi + 2\eta) + 2te^{2t} \cdot \xi$$

$$e^{2t}(\xi + 2\eta) + 2te^{2t} \cdot \xi = A(t \cdot e^{2t} \xi + e^{2t} \eta)$$

$$\left.\begin{aligned} \xi + 2 \cdot \eta &= A \cdot \eta \\ 2\xi &= A \cdot \xi \end{aligned}\right\} \quad \begin{aligned} (A - 2 \cdot I_n)\eta &= \xi \\ (A - 2 \cdot I_n) \cdot \xi &= 0 \end{aligned}$$

→ already T since ξ is an eigenvector w/ eigenvalue 2.

↳ span$\{\xi, \eta\}$ is a plane.
when you apply A to ξ (or η), you'll stay within that plane

↳ η → generalized eigenvector associated to eigenvalue r.
η is any vector satisfying $(A - r \cdot I_n)^m \cdot \eta = \xi$

↳ solving for η: $(A - 2 \cdot I_n) \cdot \eta = \xi$

$$\left(\begin{bmatrix}1 & -1\\ 1 & 3\end{bmatrix}-\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right)\cdot\eta=\begin{bmatrix}-1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\cdot\eta=\begin{bmatrix}-1\\ 1\end{bmatrix} \qquad \left[\begin{array}{cc|c}-1 & -1 & -1\\ 1 & 1 & 1\end{array}\right]\mapsto\left[\begin{array}{cc|c}1 & 1 & 1\\ 0 & 0 & 0\end{array}\right] \to \vec{\eta}=\begin{bmatrix}1-\eta_2\\ \eta_2\end{bmatrix}$$

$$\vec{\eta}=\begin{bmatrix}1\\ 0\end{bmatrix}+\eta_2\begin{bmatrix}-1\\ 1\end{bmatrix}$$

notice how this is ξ, so we can get rid of it (won't contribute anything new in our $\vec{x}^2(t)$)

$$\vec{\eta}=\begin{bmatrix}1\\ 0\end{bmatrix}$$

thus:

$$\vec{x}^2(t)=t\cdot e^{2t}\xi+e^{2t}\eta=t\cdot e^{2t}\begin{bmatrix}-1\\ 1\end{bmatrix}+e^{2t}\begin{bmatrix}1\\ 0\end{bmatrix}$$

↳ general solution: $\vec{x}(t)=C_1\cdot\vec{x}^1(t)+C_2\cdot\vec{x}^2(t)$

$$\vec{x}(t)=c_1\cdot e^{2t}\cdot\begin{bmatrix}-1\\ 1\end{bmatrix}+c_2\left(t e^{2t}\begin{bmatrix}-1\\ 1\end{bmatrix}+e^{2t}\begin{bmatrix}1\\ 0\end{bmatrix}\right)$$

↳ fundamental matrix:

$$\Phi(t)=\begin{bmatrix}-e^{2t} & -te^{2t}+e^{2t}\\ e^{2t} & te^{2t}\end{bmatrix}=e^{2t}\begin{bmatrix}-1 & 1-t\\ 1 & t\end{bmatrix}$$

↳ boundary condition: $\Phi(0) = I_n$:

$$\left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \quad \begin{array}{ll} c_{11} = 0 & c_{12} = 1 \\ c_{21} = 1 & c_{22} = 1 \end{array}$$

$$\underbrace{\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}}_{C}$$

1) $\tilde{\vec{x}}^1(t) = c_{11} \cdot \vec{x}^1 + c_{12} \cdot \vec{x}^2$:

$$\tilde{\vec{x}}^1(t) = t \cdot e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{2t} \begin{bmatrix} 1-t \\ t \end{bmatrix}$$

2) $\tilde{\vec{x}}^2(t) = c_{21} \cdot \vec{x}^1 + c_{22} \cdot \vec{x}^2$:

$$\tilde{\vec{x}}^2(t) = e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + t \cdot e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{2t} \begin{bmatrix} -t \\ 1+t \end{bmatrix}$$

↳ particular solution: $\Phi(t) = e^{2t} \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}$

- change of coordinates for homogeneous systems

↳ to solve $\vec{x}' = A\vec{x}$, we find a linear transformation $\vec{x} = T\vec{y}$ such that the system becomes diagonal.

$\vec{x} = T\vec{y}$, $\vec{x}' = T\vec{y}'$

↳ in new $\vec{y}$-coordinate : $\vec{x}' = A\vec{x}$ becomes $T\vec{y}' = AT\vec{y}$

$T\vec{y}' = AT\vec{y} \implies T^{-1}T\vec{y}' = T^{-1}AT\vec{y}$ thus:

$\vec{y}' = T^{-1}AT \cdot \vec{y}$ → we want $T^{-1}AT$ to be diagonal :

$D = T^{-1}AT \implies \vec{y}' = D \cdot \vec{y}$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

↳ here, λ_1 and λ_2 are our OG eigenvalues of A.
these eigenvalues correspond to eigenvectors $\vec{e}_1$ and $\vec{e}_2$ (elementary vectors), since the matrix is D.

→ solution in the $\vec{y}$- coordinate system:

$$\vec{y}(t) = c_1 \cdot e^{\lambda_1 t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \cdot e^{\lambda_2 t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

→ back in the $\vec{x}$- coordinate system:

↳ since $\vec{x} = T \cdot \vec{y}$:

$$\vec{x}(t) = C_1 \cdot e^{\lambda_1 t} \underbrace{\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}}_{T} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \cdot e^{\lambda_2 t} \underbrace{\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}}_{T} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x}(t) = C_1 \cdot e^{\lambda_1 t} \underset{\downarrow \atop \xi^1}{\begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix}} + C_2 \cdot e^{\lambda_2 t} \underset{\downarrow \atop \xi^2}{\begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}}$$

→ these are eigenvectors of A

↳ thus, the $\vec{x}$- coordinate system has the same eigenvalues, λ_1 and λ_2 as in $\vec{y}$, but different eigenvectors.

↳ when making T, make it:

$$T = \begin{bmatrix} \xi^1 & \xi^2 \end{bmatrix}$$

◦ the phase plane ~ linear systems

- motivation: qualitative inspection of systems (since many DFQs can't be solved analytically)
- questions about the stability of a solution.

- solution of a system $\vec{x}' = A\vec{x}$, $\vec{x} = \Phi(t)$, is a vector function that can be seen as a parametric curve, which represents the trajectory of a moving particle (whose velocity is $\vec{x}'$)

- the x_1x_2-plane is called the phase plane, and the corresponding set of trajectories is called a phase portrait.

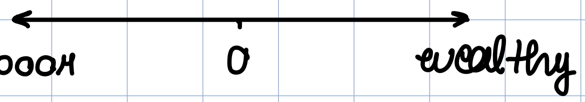
- given the system $\vec{x}' = A\vec{x}$, you get a unique trajectory on the graph, given different initial conditions (which give you c's)

- for example: if, for $\vec{x}' = A\vec{x}$, the initial condition was $\vec{x}(0) = <0, 0>$ (you start at the origin), then: for any A, you'd just stay at the origin. why? $\because$ the $\vec{x}'$ tells you how you move, and any $A \cdot <0,0> = <0, 0>$ (the origin)
- thus: $\vec{x} = <0, 0>$ is a fixed / critical point $\forall t$ and $\forall A$.

- in general: points $\vec{x}$ where $\vec{x}' = 0$ are called critical pts. and they correspond to equilibrium / constant solutions.

eigenvalues of A		type of critical point	stability
$r_1 > r_2 > 0$		node	unstable
$r_1 < r_2 < 0$		node	asymptotically stable
$r_2 < 0 < r_1$		saddle point	unstable
$r \pm i\omega$	$r < 0$	spiral point	asymptotically stable
	$r > 0$	spiral point	unstable
	$r = 0$	central point	
$r_1 = r_2 > 0$		proper/improper node	unstable
$r_1 = r_2 < 0$		proper/improper node	asymptotically stable

depending on eigenvectors



wealth → $\vec{x}$;
earning → $\vec{x}'$
$\vec{x}' > 0$ → earning ; $\vec{x}' < 0$ → spending

$\vec{x}' = A\vec{x}$:

$$\vec{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_1' = ax_1 + bx_2$ → person 1's spending/earning

$x_2' = cx_1 + dx_2$ → person 2's spending/earning

↳ if $a > 0$: "if I'm rich, I get richer;
if I'm poor, I get poorer" reinforcing feedback loop Ⓡ

↳ if $a < 0$: "if I'm rich, I spend more;
if I'm poor, I spend less." balancing feedback loop Ⓑ

↳ if $b > 0$: "if the other person is wealthy, I get wealthier;
if the other person is poor, I get poorer." Ⓡ

↳ if $b < 0$: "if the other person is wealthy, I get poorer;
if the other person is poor, I get wealthier." Ⓑ

↳ so far, our matrices were only constant (not changing)

↳ now, we consider a matrix s.t. it "depends" on its state

- autonomous systems

↳ the system doesn't depend on time (t is the independent var)

$$\frac{dx_1}{dt} = F(x_1, x_2) \qquad \frac{dx_2}{dt} = G(x_1, x_2)$$

↳ this time, A is not constant:

$$\vec{x}\,' = \begin{bmatrix} F(x_1, x_2) \\ G(x_1, x_2) \end{bmatrix} \cdot \vec{x} \qquad \text{or} \quad f(\vec{x}) = \begin{bmatrix} F(\vec{x}) \\ G(\vec{x}) \end{bmatrix} \quad f: \mathbb{R}^2 \to \mathbb{R}^2$$

↳ critical point: any point (x_1, x_2) s.t. $f(\vec{x}) = \vec{0}$; i.e. $F(x_1, x_2) = 0$ and $G(x_1, x_2) = 0$

↳ critical points $\vec{x}^0$ correspond to constant/equilibrium solutions of the system. we talked about the stability/instability/asymptotic stability of these $\vec{x}^0$ s.

① $F(x_1, x_2) = 3x_1 - 2x_2$; $G(x_1, x_2) = -x_1$

↳ this is what we've been doing so far : this is a linear, homogeneous system w/ constant coefficients :

$$\vec{x}' = \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix} \vec{x}$$

→ critical points:

↳ since A is an invertible matrix ($\det(A) \neq 0$) → it will only have one critical point → the origin ($\because$ rank$(A) = 2$)

↳ $\vec{x} = \vec{0}$ is always a critical point

↳ type of critical point:

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix} \qquad \det(A - r \cdot I_n) = (3-r)(-r) - 2 = r^2 - 3r - 2 \text{ , so:}$$

$$r = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2} \qquad r_1 > 0 \text{ , } r_2 < 0$$

↳ and then you find the eigenvectors... turns out there's two lin. independent

↳ since the eigenvalues are of the opposite sign and $\mathbb{R}$, the origin is a saddle point → unstable

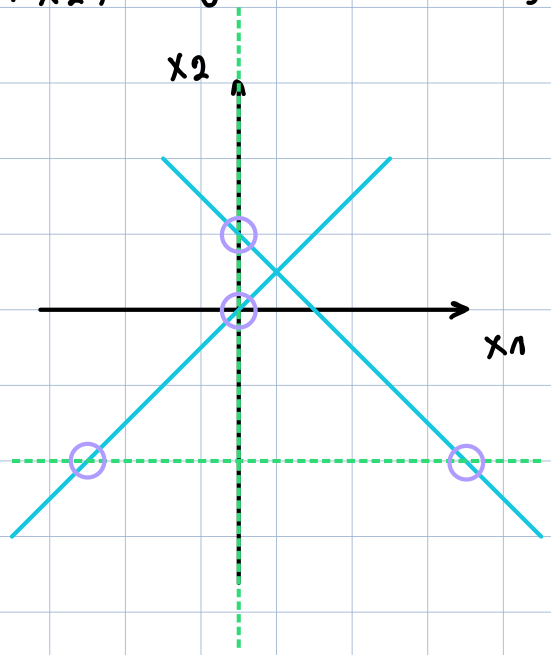
② $F(x_1, x_2) = -(x_1 - x_2)(1 - x_1 - x_2)$
$G(x_1, x_2) = x_1(2 + x_2)$

↳ this is no longer a linear system

→ critical points : find all the points (x_1, x_2) s.t. $f(\vec{x}) = \vec{0}$:

$$\left.\begin{array}{l} -(x_1 - x_2)(1 - x_1 - x_2) = 0 \\ x_1(2 + x_2) = 0 \end{array}\right\}$$

$x_1 = x_2$ OR $x_1 + x_2 = 1$ 🔵
$x_1 = 0$ OR $x_2 = -2$ 🟢



↳ critical points:

1) $(0, 0)$ → saddle
2) $(0, 1)$ → spiral
3) $(-2, -2)$ → node
4) $(3, -2)$ → node

↳ the blue and the green must intersect ($F(\vec{x}) = 0$ & $G(\vec{x}) = 0$)

→ classifying critical points:

$$F(x_1, x_2) = x_1^2 - x_1 - x_2^2 + x_2 \qquad J = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} \\ \frac{\partial G}{\partial x_1} & \frac{\partial G}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 1 & -2x_2 + 1 \\ x_2 + 2 & x_1 \end{bmatrix}$$

$$G(x_1, x_2) = 2x_1 + x_1 x_2$$

$$\frac{\partial F}{\partial x_1} = 2x_1 - 1 \qquad \frac{\partial F}{\partial x_2} = -2x_2 + 1 \qquad \frac{\partial G}{\partial x_1} = 2 + x_2 \qquad \frac{\partial G}{\partial x_2} = x_1$$

1) $\vec{x}^0 = (0,0)$; near $\vec{x}^0$: $\vec{x}' = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \vec{x}$ $\quad p = -1$, $q = -2$ $\quad r_1 = 1$, $r_2 = -2$

(0,0) is a saddle point (unstable)

2) $\vec{x}^0 = (0,1)$; near $\vec{x}^0$: $\vec{x}' = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix} \vec{x}$ $\quad p = -1$, $q = 3$ $\quad \Delta = p^2 - 4q$, $\Delta = -2$

(0,1) is a spiral point, asymptotically stable

3) $\vec{x}^0 = (-2,-2)$; near $\vec{x}^0$: $\vec{x}' = \begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix} \vec{x}$ $\quad p = -7$, $q = 10$ $\quad \Delta = p^2 - 4q$, $\Delta = 39$

(-2,-2) is a node, the asymptotically stable one

4) $\vec{x}^0 = (3,-2)$; near $\vec{x}^0$: $\vec{x}' = \begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix} \vec{x}$ $\quad p = 8$, $q = 15$ $\quad \Delta = p^2 - 4q$, $\Delta = 4$

(3,-2) is a node, the unstable one

1) case : $g : \mathbb{R} \to \mathbb{R}$:

↳ linear approximation of g near $a \in \mathbb{R}$:

$$g'(a) \approx \frac{g(x)-g(a)}{x-a} \implies g(x) \approx \underbrace{g(a) + g'(a)(x-a)}_{\text{linear}}$$

↳ if $g(a) = 0 \implies g(x) \approx g'(a)(x-a)$

→ the Calc. quote: if you zoom in on a curve, you see a line, and the slope of that line is its derivative

↳ case 2): $F : \mathbb{R}^2 \to \mathbb{R}$ (vectors to numbers)

↳ linear approximation of F near $(a,b) \in \mathbb{R}^2$:

$$F(x_1, x_2) \approx F(a, b) + \frac{\partial F}{\partial x_1}(a, b)(x_1 - a) + \frac{\partial F}{\partial x_2}(a, b)(x_2 - b)$$

↳ again, the RHS is linear

3) case: $f: \mathbb{R}^2 \to \mathbb{R}^2$ (vector to vector) $f(\vec{x}) = \begin{bmatrix} F(\vec{x}) \\ G(\vec{x}) \end{bmatrix}$

↳ linear approximation of f near $(a,b) \in \mathbb{R}^2$; for $f(a,b) = \vec{0}$:

$$f(x_1, x_2) = \begin{bmatrix} F(\vec{x}) \\ G(\vec{x}) \end{bmatrix} \approx \begin{bmatrix} \frac{\partial F}{\partial x_1}(a,b)(x_1 - a) + \frac{\partial F}{\partial x_2}(a,b)(x_2 - b) \\ \frac{\partial G}{\partial x_1}(a,b)(x_1 - a) + \frac{\partial G}{\partial x_2}(a,b)(x_2 - b) \end{bmatrix}$$

$$\approx \begin{bmatrix} \frac{\partial F}{\partial x_1}(a,b) & \frac{\partial F}{\partial x_2}(a,b) \\ \frac{\partial G}{\partial x_1}(a,b) & \frac{\partial G}{\partial x_2}(a,b) \end{bmatrix} \cdot \begin{bmatrix} x_1 - a \\ x_2 - b \end{bmatrix}$$

① consider an autonomous system: $f(\vec{x}) = \frac{d\vec{x}}{dt}$

↳ for (a,b) being a critical point (i.e. $f(a,b) = \vec{0}$)

↳ then, near a critical point (a,b), the system is approximately equal to:

$$\vec{x}' = \underbrace{\begin{bmatrix} \frac{\partial F}{\partial x_1}(a,b) & \frac{\partial F}{\partial x_2}(a,b) \\ \frac{\partial G}{\partial x_1}(a,b) & \frac{\partial G}{\partial x_2}(a,b) \end{bmatrix}}_{\text{Jacobian}} \cdot \begin{bmatrix} x_1 - a \\ x_2 - b \end{bmatrix} \quad * \text{ near } (a,b)$$

② linear approximation near critical points of the pendulum :

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -\sigma y - \omega^2 \sin(x) \end{bmatrix} \qquad \begin{aligned} F(x,y) &= y \\ G(x,y) &= -\sigma y - \omega^2 \sin(x) \end{aligned}$$

$$J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos(x) & -\sigma \end{bmatrix}$$

↳ then, near the critical point $(a,b) = (\pi, 0)$, approx :

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos(x) & -\sigma \end{bmatrix} \cdot \begin{bmatrix} x - \pi \\ y - 0 \end{bmatrix}$$

↳ change coordinate system:

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} x - \pi \\ y - 0 \end{bmatrix} \qquad \frac{d}{dt}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{bmatrix} \cdot \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \rightarrow \text{new system of ODEs}$$

↳ near $\begin{bmatrix} \pi \\ 0 \end{bmatrix} \rightsquigarrow A = \begin{bmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{bmatrix}$

$$p = tr(A) = -\gamma$$
$$q = det(A) = -\omega^2$$

↳ since $p < 0$ and $q < 0$, our critical point is a saddle point which is unstable

↳ we wanna find trajectories independent of time ("un-parameterize")

① autonomous system:

$$dx/dt = 4 - 2y$$
$$dy/dt = 12 - 3x^2$$

↳ eliminate time: divide the second eq. by the first eq.

$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{12 - 3x^2}{4 - 2y}$$ → this is separable:

$$4 - 2y \, dy = 12 - 3x^2 \, dx \implies \int 4 - 2y \, dy = \int 12 - 3x^2 \, dx$$

$$4y - y^2 = 12x - x^3 + c$$ ← there's no time involved

↳ general solution: $4y - y^2 - 12x + x^3 = C$ (just the trajectory)