Grading

Problems will be graded for accuracy and clarity of both mathematics and writing.

Problems

1. **Classification of Differential Equations**
In each of the below problems, determine the order of the given differential equation, state whether the equation is linear or nonlinear and autonomous or not. Justify your classification.

   (a) $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin t$

   (b) $(1+y^2)\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = e^t$

   (c) $\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1$

   (d) $\frac{d^2y}{dt^2} + \sin(t+y) = \sin t$

   Solution

   (a) Linear, second order, non-autonomous

   (b) Non-linear, second order, non-autonomous

   (c) linear, 4th order, autonomous

   (d) non-linear, second order, non-autonomous

2. Consider the following ODEs

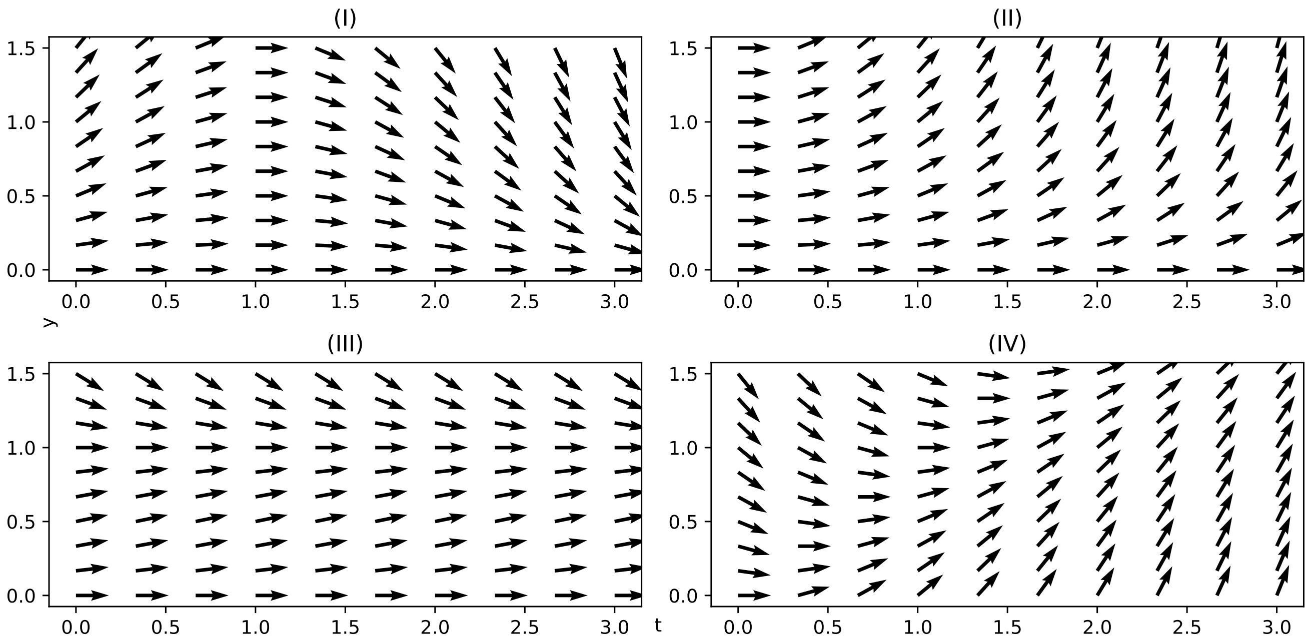
   (a) $\frac{d}{dt}y = y(1-y)$

   (b) $\frac{d}{dt}y = yt$

   (c) $\frac{d}{dt}y = t - y$

   (d) $\frac{d}{dt}y = y - ty$ [There was a sign error in the original version]

   Identify which direction field corresponds to each ODE



Solution We can immediately match (a) to (iii), since this is an autonomous ODE and (iii) only autonomous direction field. In all the others the directions change with t. Both (i) and (ii) have a $dy/dt = 0$ when $y = 0$, which is consistent with (b) and (d), but not (c), thus it must be that (c) goes with (iv). We can also notice that $dy/dt = 0$ along $y = t$ for this direction field. Now notice that in (b), $dy/dt = 0$ at $t = 0$, which is the case in (ii), but this is inconsistent with (d), hence (b) goes with (ii). To summarize:

(a) (iii)

(b) (ii)

(c) (iv)

(d) (i)

3. Find the solution to the IVP
$$\frac{d}{dt}y = y + t^2, \quad y(0) = 1$$
and describe the long-term (large t behavior of the solution)

Solution: We can rewrite the equation as
$$\frac{d}{dt}y - y = t^2$$
which has the form
$$\frac{d}{dt}y - py = g$$
with $p = 1$ and $g = t^2$ We can solve this with the integrating factor $\mu = e^{-t}$. The general solution is
$$\begin{aligned} y(t) &= \frac{C}{\mu(t)} + \frac{1}{\mu(t)} \int_0^t \mu(s)s^2 ds \\ &= Ce^t + e^t \int_0^t e^{-s}s^2 ds \end{aligned}$$
The integral can be evaluated with integrating by parts twice
$$\begin{aligned} \int_0^t e^{-s}s^2 ds &= -e^t + 2\int_0^t e^{-s} s ds \\ &= -e^{-t}t^2 - 2e^{-t}t - 2e^{-t} + 2 \\ &= 2 - e^{-t}(2 + 2t + t^2) \end{aligned}$$
Thus the general solution is
$$y(t) = Ce^t + 2e^t - (2 + 2t + t^2)$$
Using the initial condition
$$y(0) = C + 2 - 2 = C = 1$$
so the solution to the IVP is
$$y(t) = e^t + 2e^t - (2 + 2t + t^2) = 3e^t - 2 - 2t - t^2$$

4. Solve the initial value problem

$$\frac{d}{dt}y = \frac{y^2}{t}, \quad y(1) = 1$$

Solution The equation is separable. Dividing both sides by y^2 and integrating

$$\begin{aligned}\int \frac{1}{y^2}\frac{d}{dt}y dt = \int \frac{1}{t}dt \\ \implies \int \frac{1}{y^2}dy = \int \frac{1}{t}dt \\ \implies -\frac{1}{y} = \ln(t) + C \\ \implies y = -\frac{1}{\ln(t) + C}\end{aligned}$$

Using the initial condition

$$y(1) = -\frac{1}{\ln(1) + C} = -\frac{1}{C} = 1 \implies C = -1$$

therefore the solutions is

$$y(t) = \frac{1}{1 - \ln(t)}$$

5. What does the theorem on existence and uniqueness tell us about solutions to the following initial value problem?

$$\frac{d}{dt}y = \frac{1}{y(t^2-1)}, \quad y(1.03) = 1$$

Solution The right hand side $f(t,y) = 1/(y(t^2-1))$ and it partial derivative

$$\frac{\partial}{\partial}f(t,y) = -\frac{2t}{(t^2-1)^2 y}$$

is continuous in the rectangle

$$R = \{(t,y) : 1 < t < \infty, 0 < y < \infty\}$$

containing the initial point $(t_0, y_0) = (1.03, 1)$. The existence theorem tells us there exists a unique solution defined for t in some interval $I = (1.03 - h, 1.03 + h)$ where $h < 0.03$.