Grading

Problems will be graded for accuracy and clarity of both mathematics and writing.

Problems

1. (a) Find a differential equation with the general solution

$$y(t) = c_1 e^{-t} + c_2 e^{2t}.$$

(b) Provide the initial condition that leads to the solution with $c_1 = 1$ and $c_2 = 5$.

2. Consider a spring-mass system governed by the equation

$$x'' = -6x' - 13x$$

with initial state $x(0) = 0$ and $x'(0) = 4$.

(a) Find an expression for the motion of the spring-mass as a function of time t.

(c) Describe the long-term behavior of mass-spring as $t \to \infty$.

(b) At what time $0 \leq t < \infty$ is the value of $x(t)$ maximum? What is this maximum value?

3. Find the general solution to

$$y'' + 3y' - 10y = 5t^2 \tag{1}$$

4. Find the general solution to

$$y'' - 4y' + 21y = 0 \tag{2}$$

Solutions

Q1

$$y(t) = e^{-4t} + 5e^{-2t}$$

roots of charactristic eq:

$$\begin{aligned} r_1 = -4, r_2 = -2 &\Rightarrow (r+4)(r+2) = 0 \\ &\Rightarrow r^2 + 6r + 8 = 0 \\ &\Rightarrow y'' + 6y' + 8y = 0 \end{aligned}$$

initial condition:

$$y(0) = 6, y'(0) = -14$$

Q2

$$\begin{aligned} x'' + 4x' + 5x = 0 \\ x(0) = 0, x'(0) = 1 \end{aligned}$$

char. eq.

$$\begin{aligned} &r^2 + 4r + 5 = 0 \\ &\Rightarrow r = \frac{-4 \pm \sqrt{-4}}{2} \\ &\Rightarrow r_1 = -2 - i, r_2 = -2 + i \\ &x(t) = e^{-2t}\left(c_1 e^{-it} + c_2 e^{it}\right) \end{aligned}$$

real form:

$$x(t) = e^{-2t}\left(c_1 \cos t + c_2 \sin t\right)$$

given boundary conditions $c_1 = 0$ and $c_2 = 1$

$$\Rightarrow x(t) = e^{-2t} \sin(t)$$

(b) Largest distance to the origin. We use the derivative test.

$$x'(t) = -2e^{-2t}\sin(t) + e^{-2t}\cos(t)$$

if $x'(t_1) = 0$ then $2\sin(t_1) = \cos(t_1)$ and therefore $\tan(t_1) = 1/2$. Smallest such value is $t_1 = 0.46$, where the second derivative is negative.

(c) The spring tends to an equilibrium rest stale near $x = 0$.

Q3

$$y'' + y = e^{2t}$$

we use the method of undetermined coefs. complementary solution:

$$\begin{aligned} & y'' + y = 0 \\ \Rightarrow & r^2 + 1 = 0 \rightarrow r_1 = i, r_2 = -i \\ \Rightarrow & y_1(t) = e^{it}, y_2(t) = e^{-it} \end{aligned}$$

Particular Solution.

$$y'' + y = e^{2t}$$

Let $Y(t) = Ae^{2t}$,

$$\begin{aligned} & Y'(t) = 2Ae^{2t} \\ & Y''(t) = 4Ae^{2t} \\ \Rightarrow & \left(4Ae^{2t}\right) + Ae^{2A} = e^{2t} \\ \Rightarrow & 5A = 1 \Rightarrow A = \frac{1}{5} \\ \Rightarrow & Y(t) = \frac{1}{5}e^{2t} \end{aligned}$$

General solution:

$$y(t) = c_1 e^{it} + c_2 e^{-it} + \frac{1}{5}e^{2t}$$

real form:

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + \frac{1}{5}e^{zt}$$

Q4

$$9y'' + 6y' + y = 0$$

char. eq.

$$\begin{aligned} & 9r^2 + 6r + 1 = 0 \\ & \Rightarrow r = \frac{-6 \pm \sqrt{0}}{18} \\ & \Rightarrow r_1 = -\frac{1}{3}, r_2 = -\frac{1}{3} \\ & (\text{ repeated root}) \\ & y_1(t) = e^{-\frac{t}{3}} \\ & y_2(t) = te^{-t/3} \end{aligned}$$

general solution: $y(t) = c_1 e^{-t/3} + c_2 t e^{-t/3}$

Q5

$$x' = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} x$$
$$x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Eigenvalues:

$$\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (1-\lambda)^2 + i^2 = 0$$
$$\Rightarrow \lambda^2 - 2\lambda = 0$$
$$\Rightarrow \lambda_1 = 0, \lambda_2 = 2$$

Eigenvector of λ_1:

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\xi^{(1)} = \begin{bmatrix} i \\ -1 \end{bmatrix}$$

Eigenvector of λ_2:

$$\begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\xi^{(2)} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

general solution:

$$x(t) = c_1 \begin{bmatrix} i \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

using the boundary condition

$$x(0) = c_1 \begin{bmatrix} i \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

we can take $c_1 = c_2 = -i$ and

$$x(t) = \begin{bmatrix} e^{2t} + 1 \\ -ie^{2t} + i \end{bmatrix}$$