

Homework 1

Metric Spaces and Real Analysis

Instructions. Write complete, rigorous proofs. State clearly which definitions and theorems you use. Throughout, $\mathbb{R}$ carries the absolute-value metric $d(x, y) = |x - y|$.

Brief review: supremum and infimum

Let $E \subseteq \mathbb{R}$ be nonempty and bounded above. The *supremum* $\sup E$ is the *least upper bound* of E ; concretely, $s = \sup E$ means

- (i) $x \leq s$ for every $x \in E$ (so s is an upper bound), and
- (ii) for every $\varepsilon > 0$ there exists $x \in E$ with $x > s - \varepsilon$ (no smaller number is an upper bound).

The *infimum* $\inf E$ is defined dually as the greatest lower bound. The completeness of $\mathbb{R}$ guarantees these exist for nonempty sets that are, respectively, bounded above or below. Note that the supremum need not belong to E : for example $\sup(0, 1] = 1$ (attained) while $\inf(0, 1] = 0$ (not attained).

Problems

P1. Regard $\mathbb{R}$ as a metric space with the absolute-value metric. Let $E \subseteq \mathbb{R}$ be nonempty with $s = \sup E < \infty$. Prove that there exists a sequence in E that converges to s .

P2. Let (X, d) be a metric space and fix a point $x_0 \in X$. Define $f : X \rightarrow \mathbb{R}$ by

$$f(x) = d(x, x_0).$$

Prove that f is continuous.

P3. Prove that every convergent sequence in a metric space is a Cauchy sequence.

P4. Let (X, d) be a metric space, and let $U_1, \dots, U_m$ be balls of radius 1 that cover X , i.e. $X \subseteq \bigcup_{j=1}^m U_j$. Show that every sequence in X has a subsequence all of whose terms lie in a single one of the balls U_j .

P5. (a) Prove that every sequence of real numbers has a monotone subsequence (that is, a subsequence that is either nondecreasing or nonincreasing).

(b) Using part (a), prove the *Bolzano–Weierstrass theorem* in $\mathbb{R}$ (every bounded sequence of real numbers has a convergent subsequence).