

Homework 2

Metric Spaces and Real Analysis

Reminders

For a nonempty subset A of a metric space (X, d) , the *diameter* is $\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}$. A metric space is *compact* if every open cover has a finite subcover. In a metric space this is equivalent to every sequence having a convergent subsequence. The *open ball* of radius $r > 0$ about x is $B(x, r) = \{y : d(x, y) < r\}$, and the *closed ball* is $\bar{B}(x, r) = \{y : d(x, y) \leq r\}$.

Problems

P1. Let $X = \mathbb{R}^n$ with the metric

$$d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|, \quad x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

- (a) Describe a ball of radius $r > 0$ in X , and identify its shape.
- (b) Let M be the smallest number of closed balls of radius $r > 0$ needed to cover a closed ball of radius $2r$. Find a formula relating M to the dimension n of $\mathbb{R}^n$. (This observation motivates the definition of the *doubling dimension* of a metric space, which is phrased using open balls of radii r and $2r$.)

P2. Let $X = \ell^\infty$ be the metric space of all bounded real sequences $x = (x_1, x_2, \dots)$ equipped with the ℓ^∞ distance

$$d(x, y) = \sup_i |x_i - y_i|, \quad x, y \in X.$$

Prove that the closed ball $B = \{x \in X : \sup_i |x_i| \leq 1\}$ is not compact.

P3. Let X be a *complete* metric space, and let $\mathcal{I}$ be the collection of all finite binary strings of length ≥ 1 . Suppose that to each string $(i_1, \dots, i_m) \in \mathcal{I}$ there is assigned a subset $A_{i_1, \dots, i_m} \subseteq X$ such that

- (i) $X \subseteq A_0 \cup A_1$
- (ii) $A_{i_1, \dots, i_m} \subseteq A_{i_1, \dots, i_m, 0} \cup A_{i_1, \dots, i_m, 1}$ for every string $(i_1, \dots, i_m) \in \mathcal{I}$
- (iii) for every $\varepsilon > 0$ there exists N such that $\text{diam}(A_{i_1, \dots, i_n}) < \varepsilon$ for every binary string $(i_1, \dots, i_n)$ of length $n > N$

Prove that X is compact.

P4. Using P3, prove that $[0, 1] \times [0, 1]$ is compact in $\mathbb{R}^2$ with the Euclidean metric.

P5. Fix a constant q with $0 < q \leq \frac{1}{2}$. For $\alpha \in [0, 1]$, define a function f_α on the real line by

$$f_\alpha(x) = \begin{cases} \frac{1}{2q}, & \alpha - q < x < \alpha + q, \\ 0, & \text{otherwise.} \end{cases}$$

Let $X = \{f_\alpha : \alpha \in [0, 1]\}$ with the metric

$$\rho(f_\alpha, f_\beta) = \int_0^1 |f_\alpha(x) - f_\beta(x)| dx.$$

- (a) Show that (X, ρ) is a metric space.
- (b) Prove that X is compact.