

Homework 3

Metric Spaces and Real Analysis

Problems

- P1.** Define what a *separable* metric space is, and give an example of a metric space that is not separable.
- P2.** Give an example of a metric space that is bounded but not totally bounded. (Here *bounded* means that there exists $M > 0$ with $d(x, y) < M$ for all $x, y \in X$.)
- P3.** Consider the five attributes of a metric space

(1) complete, (2) compact, (3) bounded, (4) totally bounded, (5) separable.

A binary string $b_1b_2b_3b_4b_5$ of length 5 encodes a combination of these attributes, where $b_k = 1$ means the space has attribute (k) and $b_k = 0$ means it does not. For example, 10001 designates a metric space that is complete and separable but is neither compact, totally bounded, nor bounded.

Flip a coin to generate four random binary strings of length 5. For each string, either give an example of a metric space with exactly that combination of attributes, or prove, using the theorems from this chapter, that no such space exists.

- P4.** Write down a sequence $\{f_n\}$ of real-valued functions on the closed interval $[0, 1]$ such that
- (i) $\{f_n\}$ is uniformly bounded,
 - (ii) $\{f_n\}$ is not equicontinuous, and
 - (iii) $\{f_n\}$ has no subsequence that converges uniformly to a continuous function on $[0, 1]$.
- P5.** (*Optional.*) Let (X, d) be a complete metric space, and let $\mathcal{K}(X)$ be the set of all nonempty compact subsets of X . For $A, B \in \mathcal{K}(X)$, define the *Hausdorff distance*

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}.$$

You may assume that $(\mathcal{K}(X), d_H)$ is a complete metric space.

Suppose $f_1, f_2: X \rightarrow X$ satisfy

$$d(f_i(x), f_i(y)) \leq c d(x, y), \quad i = 1, 2,$$

for some $0 < c < 1$, and define $T(A) = f_1(A) \cup f_2(A)$, where $f_i(A) = \{f_i(a) : a \in A\}$. Show that

$$d_H(T(A), T(B)) \leq c d_H(A, B) \quad \text{for all } A, B \in \mathcal{K}(X),$$

and use the Banach contraction principle to conclude that there is a unique nonempty compact set $K \subseteq X$ satisfying $K = f_1(K) \cup f_2(K)$.

(*Optional.*) As an example, take $X = \mathbb{R}$ and

$$f_1(x) = \frac{x}{3}, \quad f_2(x) = \frac{x}{3} + \frac{2}{3}.$$

What familiar compact set satisfies $K = f_1(K) \cup f_2(K)$?