

Review Problem Set

Midterm 1

Problems

Problem 1. Let A and B be nonempty subsets of a metric space (X, d) , and let

$$c = \inf\{d(x, y) : x \in A, y \in B\}.$$

- (a) Prove that there exist sequences $\{x_n\} \subseteq A$ and $\{y_n\} \subseteq B$ such that $\lim_{n \rightarrow \infty} d(x_n, y_n) = c$.
- (b) Prove that if A and B are compact, then there exist $x_0 \in A$ and $y_0 \in B$ such that $d(x_0, y_0) = c$.

Problem 2. Let (X, d) be a complete metric space, fix $x_0 \in X$, and define

$$S = \{x \in X : d(x, x_0) = 1\}.$$

Is it true that every sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq S$ has a convergent subsequence? Prove it, or disprove it with a counterexample.

Problem 3. Let X be the space of solid closed unit squares in $[0, 10] \times [0, 10] \subseteq \mathbb{R}^2$ of the form

$$X = \{[a, a+1] \times [b, b+1] : 0 \leq a \leq 9, 0 \leq b \leq 9\}.$$

For $P, Q \in X$ define

$$d(P, Q) = \text{Area}((P \setminus Q) \cup (Q \setminus P)),$$

the area of the symmetric difference $P \triangle Q$. Prove that (X, d) is a metric space and that (X, d) is compact. (Draw a picture for this problem.)

Problem 4. Find metrics d and ρ on $\mathbb{R}$ with the following property: a map $f : (\mathbb{R}, d) \rightarrow (\mathbb{R}, \rho)$ is continuous *if and only if* f is constant on $(-\infty, 0)$. (There is no restriction on f on $[0, \infty)$; there it may be an arbitrary function.) Justify your answer.

Problem 5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be measurable with respect to the Borel σ -algebra, and let $A \subseteq \mathbb{R}$ be a Borel set. Define

$$g(x) = \begin{cases} f(x), & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Prove directly from the definitions that $g^{-1}(A)$ is a Borel set.

Problem 6. For $\alpha, \beta \in \mathbb{R}$ let

$$A_\alpha = \{(x, y) \in \mathbb{R}^2 : x \leq \alpha\}, \quad B_\beta = \{(x, y) \in \mathbb{R}^2 : y \leq \beta\},$$

and let $\mathcal{F}$ be the σ -algebra on $\mathbb{R}^2$ generated by all the sets A_α , $\alpha \in \mathbb{R}$, and B_β , $\beta \in \mathbb{R}$. Which of the following sets belong to $\mathcal{F}$? Justify using a direct argument.

- (i) $S = \{(x, y) \in \mathbb{R}^2 : \alpha_1 < x \leq \alpha_2, \beta_1 < y \leq \beta_2\}$, where $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$ are real numbers.
- (ii) $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

Problem 7. Let $X = \mathbb{R}$ and let

$$\mathcal{F} = \{A \subseteq \mathbb{R} : \text{if } a \in A, \text{ then } a + 1 \in A \text{ and } a - 1 \in A\}.$$

- (a) Show that $\mathcal{F}$ is a σ -algebra on $\mathbb{R}$.
- (b) Give a full description of all measurable functions $f : (X, \mathcal{F}) \rightarrow \mathbb{R}$, where $\mathbb{R}$ is equipped with the Borel σ -algebra.

Solutions

Solution 1

(a) Since $A, B \neq \emptyset$, c is a real number ≥ 0 . By the definition of infimum, for each $n \in \mathbb{N}$ the number $c + \frac{1}{n}$ is not a lower bound, so there are $x_n \in A$, $y_n \in B$ with

$$c \leq d(x_n, y_n) < c + \frac{1}{n}.$$

Hence $d(x_n, y_n) \rightarrow c$.

(b) Take the sequences from (a). Since A is compact, some subsequence $x_{n_k} \rightarrow x_0 \in A$. Since B is compact, the sequence $\{y_{n_k}\}$ has a further subsequence $y_{n_{k_j}} \rightarrow y_0 \in B$, and still $x_{n_{k_j}} \rightarrow x_0$. By the triangle inequality,

$$|d(x_n, y_n) - d(x_0, y_0)| \leq d(x_n, x_0) + d(y_n, y_0),$$

so along this subsequence $d(x_{n_{k_j}}, y_{n_{k_j}}) \rightarrow d(x_0, y_0)$. But this subsequence of real numbers also converges to c , so $d(x_0, y_0) = c$.

Solution 2

False. Let $X = \ell^2$, which is complete, let $x_0 = 0$, and let $x_n = e_n$ be the n -th standard unit vector. Then $\|e_n\| = 1$, so $e_n \in S$, but

$$\|e_n - e_m\| = \sqrt{2} \quad (n \neq m).$$

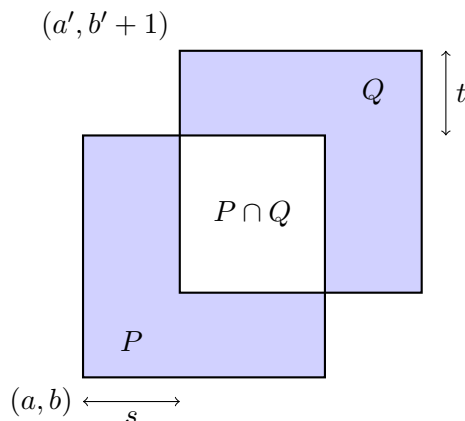
So no subsequence is Cauchy, and hence no subsequence converges.

Alternative. Let X be any infinite set with the discrete metric. It is complete, because a Cauchy sequence is eventually constant. Here $S = X \setminus \{x_0\}$, and a sequence of distinct points of S has no convergent subsequence.

Remark. In $\mathbb{R}^n$ with the Euclidean metric the statement *is* true: S is closed and bounded, hence compact by Heine–Borel.

Solution 3

Picture and formula. Write $P = [a, a+1] \times [b, b+1]$ and $Q = [a', a'+1] \times [b', b'+1]$, and set $s = |a - a'|$, $t = |b - b'|$. In the figure below, the shaded region is $P \triangle Q$.



The intersection $P \cap Q$ is a rectangle with side lengths $(1-s)^+$ and $(1-t)^+$, where $u^+ = \max\{u, 0\}$. Since $\text{Area}(P) = \text{Area}(Q) = 1$,

$$d(P, Q) = \text{Area}(P) + \text{Area}(Q) - 2 \text{Area}(P \cap Q) = 2 - 2(1-s)^+(1-t)^+.$$

d is a metric.

- *Nonnegativity and symmetry.* Both are clear, since d is an area and $P \triangle Q = Q \triangle P$.
- *Identity.* We have $d(P, P) = 0$. If $P \neq Q$, then $s > 0$ or $t > 0$, so $(1-s)^+(1-t)^+ < 1$ and therefore $d(P, Q) > 0$.
- *Triangle inequality.* We have $P \triangle R \subseteq (P \triangle Q) \cup (Q \triangle R)$. Indeed, if a point lies in exactly one of P, R , then whether or not it lies in Q , it lies in exactly one of P, Q or in exactly one of Q, R . By monotonicity and subadditivity of area,

$$d(P, R) \leq d(P, Q) + d(Q, R).$$

Compactness. Define $\varphi : [0, 9]^2 \rightarrow X$ by $\varphi(a, b) = [a, a+1] \times [b, b+1]$; it is onto by definition of X . We claim

$$d(\varphi(a, b), \varphi(a', b')) \leq 2(s+t).$$

If $s, t \leq 1$, then $d = 2(1 - (1-s)(1-t)) = 2(s+t-st) \leq 2(s+t)$. If $s > 1$ or $t > 1$, then $d = 2 < 2(s+t)$. Thus φ is continuous. Since $[0, 9]^2$ is compact by Heine–Borel and continuous images of compact sets are compact, $X = \varphi([0, 9]^2)$ is compact.

Solution 4

Answer. Let ρ be the *discrete metric* on $\mathbb{R}$, so $\rho(u, v) = 1$ for $u \neq v$ and every subset of $(\mathbb{R}, \rho)$ is open. Let

$$d(x, y) = |x - y| + \sigma(x, y), \quad \sigma(x, y) = \begin{cases} 1, & x \neq y \text{ and } \max\{x, y\} \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

d is a metric (you can check)

Topology of $(\mathbb{R}, d)$. If $x \geq 0$, then $d(x, y) \geq 1$ for $y \neq x$, so x is isolated. If $x < 0$ and $0 < r \leq \min\{|x|, 1\}$, then $B_d(x, r) = (x-r, x+r)$, the usual interval. Hence $U = (-\infty, 0)$ and $V = [0, \infty)$ are both open, d induces the usual topology on U , and the discrete topology on V .

Proof of the equivalence. Since $\mathbb{R} = U \sqcup V$ with U, V open, f is continuous iff $f|_U$ and $f|_V$ are continuous. Now $f|_V$ is always continuous because V is discrete, so there is no condition on $[0, \infty)$. Since U is connected and $(\mathbb{R}, \rho)$ is discrete, $f|_U$ is continuous iff $f(U)$ is a single point: a continuous image of a connected set is connected, and the connected subsets of a discrete space are points; conversely constant maps are continuous. Hence f is continuous iff f is constant on $(-\infty, 0)$.

Solution 5

Splitting according to the sign of x ,

$$g^{-1}(A) = (f^{-1}(A) \cap [0, \infty)) \cup \{x < 0 : 0 \in A\} = \begin{cases} (f^{-1}(A) \cap [0, \infty)) \cup (-\infty, 0), & 0 \in A, \\ f^{-1}(A) \cap [0, \infty), & 0 \notin A. \end{cases}$$

Each piece is Borel:

- $f^{-1}(A)$ is Borel because f is Borel measurable and A is Borel.
- $[0, \infty)$ is closed and $(-\infty, 0)$ is open, so both are Borel.

Finite intersections and unions of Borel sets are Borel, so $g^{-1}(A)$ is Borel in either case.

Solution 6

(i) **Yes.** We have

$$\{\alpha_1 < x \leq \alpha_2\} = A_{\alpha_2} \setminus A_{\alpha_1}, \quad \{\beta_1 < y \leq \beta_2\} = B_{\beta_2} \setminus B_{\beta_1},$$

and therefore

$$S = (A_{\alpha_2} \setminus A_{\alpha_1}) \cap (B_{\beta_2} \setminus B_{\beta_1}) \in \mathcal{F},$$

since σ -algebras are closed under differences and intersections.

(ii) **Yes.** Let $\mathcal{R}$ be the family of all rectangles $(q_1, q_2] \times (r_1, r_2]$ with $q_i, r_i \in \mathbb{Q}$ that are contained in D . It is countable, and each member lies in $\mathcal{F}$ by (i). Let $p = (p_1, p_2) \in D$. Since D is open, there is $\varepsilon > 0$ with $B(p, \varepsilon) \subseteq D$. Choose rationals with

$$p_1 - \frac{\varepsilon}{2} < q_1 < p_1 < q_2 < p_1 + \frac{\varepsilon}{2}, \quad p_2 - \frac{\varepsilon}{2} < r_1 < p_2 < r_2 < p_2 + \frac{\varepsilon}{2}.$$

Then $p \in (q_1, q_2] \times (r_1, r_2] \subseteq B(p, \varepsilon) \subseteq D$. Hence

$$D = \bigcup_{R \in \mathcal{R}} R$$

is a countable union of sets in $\mathcal{F}$, so $D \in \mathcal{F}$. The same argument shows every open set, hence every Borel set, of $\mathbb{R}^2$ lies in $\mathcal{F}$; since each A_α, B_β is closed, $\mathcal{F}$ equals the Borel σ -algebra of $\mathbb{R}^2$.

Solution 7

First, a reformulation. If $A \in \mathcal{F}$ and $a \in A$, applying the condition repeatedly gives $a + n \in A$ for every $n \in \mathbb{Z}$. Thus

$$A \in \mathcal{F} \iff A + \mathbb{Z} = A \iff A \text{ is a union of cosets } x + \mathbb{Z}.$$

(a) We check the three axioms.

- $\mathbb{R} \in \mathcal{F}$. The condition holds trivially, since $a \pm 1 \in \mathbb{R}$ for every a . (Also $\emptyset \in \mathcal{F}$ vacuously.)
- *Complements.* Let $A \in \mathcal{F}$ and $a \in \mathbb{R} \setminus A$. If $a + 1 \in A$, then $(a + 1) - 1 = a \in A$, a contradiction; so $a + 1 \in \mathbb{R} \setminus A$. Similarly, if $a - 1 \in A$, then $(a - 1) + 1 = a \in A$, a contradiction; so $a - 1 \in \mathbb{R} \setminus A$. Hence $\mathbb{R} \setminus A \in \mathcal{F}$.
- *Countable unions.* Let $A_1, A_2, \dots \in \mathcal{F}$ and $a \in \bigcup_k A_k$. Then $a \in A_k$ for some k , so $a \pm 1 \in A_k \subseteq \bigcup_k A_k$. Hence $\bigcup_k A_k \in \mathcal{F}$.

So $\mathcal{F}$ is a σ -algebra. (In fact it is closed under *arbitrary* unions and intersections.)

(b) **Answer.** f is $\mathcal{F}$ -measurable if and only if f is 1-periodic, i.e. $f(x + 1) = f(x)$ for all $x \in \mathbb{R}$. No further regularity is needed.

($\Rightarrow$) Let $x \in \mathbb{R}$ and $t = f(x)$. The singleton $\{t\}$ is Borel, so $E = f^{-1}(\{t\}) \in \mathcal{F}$. Since $x \in E$, also $x + 1 \in E$, i.e. $f(x + 1) = t = f(x)$.

($\Leftarrow$) Let f be 1-periodic and let $B \subseteq \mathbb{R}$ be Borel (in fact any subset works). If $a \in f^{-1}(B)$, then periodicity gives $f(a+1) = f(a) \in B$, and periodicity applied at $a-1$ gives $f(a-1) = f(a) \in B$. So $a \pm 1 \in f^{-1}(B)$. Hence $f^{-1}(B) \in \mathcal{F}$.

Equivalently, the measurable functions are exactly $f = h \circ \pi$, where $\pi(x) = x - \lfloor x \rfloor$ and $h : [0, 1) \rightarrow \mathbb{R}$ is *arbitrary*.

Examples. The functions $\sin(2\pi x)$, $\mathbf{1}_{\mathbb{Q}}$, $\mathbf{1}_{\mathbb{Z}}$, and $x - \lfloor x \rfloor$ are all measurable. Neither $\mathbf{1}_{[0, \infty)}$ nor $f(x) = x$ is measurable, because neither is periodic.