

Group Discussion #3

All functions below are real-valued.

Warm-up

Review, as a group, the following two definitions.

- (1) A sequence $\{f_n\}$ of functions on a set E is **uniformly bounded** if there is a constant $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in E$ and all n .
- (2) A sequence $\{f_n\}$ of functions on a metric space (E, d) is **equicontinuous** if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ whenever $d(x, y) < \delta$, for all $x, y \in E$ and all n .

Problems

Q1. Given real constants $-M \leq a, b, c \leq M$, define

$$f(x) = \begin{cases} a, & 0 \leq x \leq 1, \\ b, & 2 \leq x \leq 3, \\ c, & 4 \leq x \leq 5, \end{cases}$$

and extend f to the unique continuous real-valued function on $[0, 5]$ that is piecewise linear on each integer subinterval $[k, k+1]$. Let X be the space of all such functions on $[0, 5]$ with the supremum norm, where M is a positive constant. Prove the following two statements.

- (a) Every sequence of functions in X is equicontinuous and uniformly bounded.
- (b) Every sequence of functions in X has a convergent subsequence.

Q2. Write down a sequence of functions $\{f_n\}$ on $[0, 1]$ such that

- (i) $\{f_n\}$ is uniformly bounded
- (ii) $\{f_n\}$ is **not** equicontinuous
- (iii) no subsequence of $\{f_n\}$ converges to a continuous function on $[0, 1]$

Q3. (Optional) Write down a sequence of functions $\{f_n\}$ on $[0, 1]$ such that

- (i) $\{f_n\}$ is **not** uniformly bounded
- (ii) $\{f_n\}$ is equicontinuous
- (iii) no subsequence of $\{f_n\}$ converges to a continuous function on $[0, 1]$