Homework 1

Math 76 Summer 2026

Name: ______________________________

Problem 1

For each $i = 1, 2, 3$, define the open half-space

$$A_i = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{u}_i^T \mathbf{x} + c_i > 0\}$$

and the sign classifier

$$f_i(\mathbf{x}) = \operatorname{sign}(\mathbf{u}_i^T \mathbf{x} + c_i) \in \{-1, 0, +1\}.$$

(a) In words, explain the meaning of $f_i(\mathbf{x}) = +1$, $f_i(\mathbf{x}) = 0$, and $f_i(\mathbf{x}) = -1$.

(b) Compute the sign vector

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \end{bmatrix}$$

at the following test points:

$$Q_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} 2 \\ \frac{3}{2} \end{bmatrix},$$

$$Q_5 = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \quad Q_6 = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}, \quad Q_7 = \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}.$$

Point	$f_1(\mathbf{x})$	$f_2(\mathbf{x})$	$f_3(\mathbf{x})$	$\mathbf{f}(\mathbf{x})$
Q_1				
Q_2				
Q_3				
Q_4				
Q_5				
Q_6				
Q_7				

(c) The three lines divide the plane into 7 open regions. Explain why two points in the same open region must have the same sign vector.

Problem 2

Using the same classifiers f_1, f_2, f_3 from Problem 2, define

$$F(\mathbf{x}) = \mathbf{w}^T \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \end{bmatrix}, \qquad \mathbf{w} \in \mathbb{R}^3.$$

Ignore boundary points where one of the classifiers is equal to 0.

(a) Let $\mathbf{w} = (-1, -1, 1)^T$. Find all points among $Q_1, \ldots, Q_7$ from Problem 2 where $F(\mathbf{x}) = +1$.

(b) Describe the set $\{\mathbf{x} \in \mathbb{R}^2 : F(\mathbf{x}) = +3\}$ geometrically. Your answer should use the half-spaces.

(c) Let $\mathbf{w} = (1, 1, 1)^T$. What possible values can $F(\mathbf{x})$ take away from the boundary lines? Compute $F(Q_j)$ for $j = 1, \ldots, 7$ using your table from Problem 2.

Problem 3

Consider the space of all functions $f : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$f(x) = \sigma(\mathbf{u}^T x + b), \qquad \mathbf{u} \in \mathbb{R}^2, \quad b \in \mathbb{R},$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is the Heaviside function

$$\sigma(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0. \end{cases}$$

(i) Describe the space of all such functions geometrically.

(ii) If

$$g(x_1, x_2) = \begin{cases} 1, & x_2 \leq 3, \\ 0, & x_2 > 3, \end{cases}$$

can you find $\mathbf{u} \in \mathbb{R}^2$ and $b \in \mathbb{R}$ so that

$$g(x) = \sigma(\mathbf{u}^T x + b)?$$

Problem 4

In class, we saw an example where three lines in general position divide $\mathbb{R}^2$ into 7 open regions.

(a) Suppose one line divides the plane into 2 regions. Add a second line that is not parallel to the first. Explain why the number of regions becomes 4.

(b) Add a third line which is not parallel to either previous line and does not pass through their intersection point. Explain why the third line cuts through three existing regions and therefore creates three new regions. How many regions we have in general? (exclude rare cases when lines have special structures)

(c) More generally, guess a formula for the maximum number of regions created by m lines in the plane. Briefly justify your formula.

Problem 5

Consider a neural network with 2 input layers, 2 hidden layers and 2 output layers (three layers total), using the sign function as the activation function . More explicitly,

$$f(x) = \sigma\left(A(\sigma(Bx + c) + b\right)$$

where A and B are 2×2 matrices and $b, c \in \mathbb{R}^2$. Here, the activation function σ turns a 2D vector of real numbers into a 2D vector of ± 1, and 0's. For instance, if $y = \begin{bmatrix} -2 \\ 3.5 \end{bmatrix}$ then the activation function applies the sign function to each component of y, and $\sigma(y) = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$.

(a) Describe both layers of this neural network using the row picture.

(b) What are the possible values of nodes in the hidden layer (these are finitely many points in 2D).

(c) The last layer in the row picture comes with two choices of half spaces. Explain briefly how positions of those half spaces relative to possible values of the points in part (b) lead to the value of the output node. Use two examples for illustration.