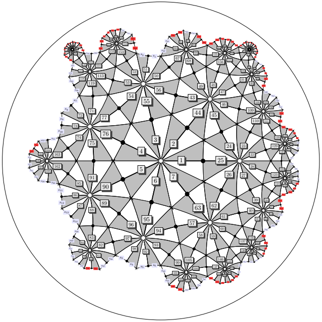
Computing a Database of Belyĭ Maps



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Acknowledgements

This is joint work with:

- Mike Klug
- Sam Schiavone
- Jeroen Sijsling
- John Voight

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We can view φ as a branched (ramified) covering map of Riemann surfaces...

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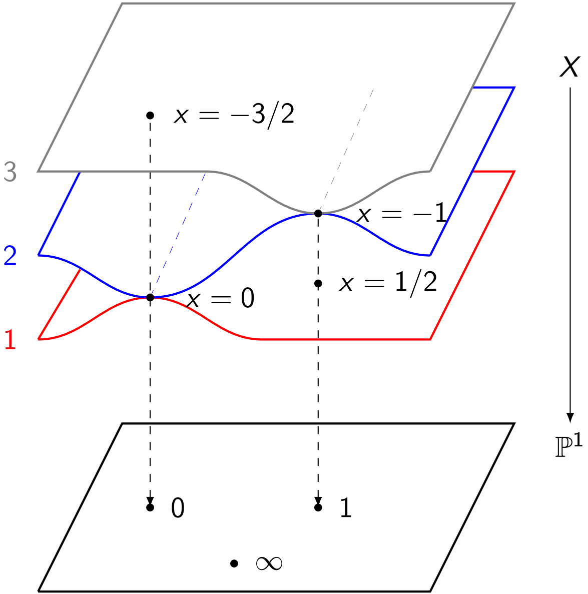

$$\varphi(x) = \underbrace{2x^3 + 3x^2}_{z}$$

Riemann's Existence Theorem

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A consequence of this theorem is the equivalence between Riemann surfaces and algebraic curves.

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We say X is **defined over** a subfield L of $\mathbb{C}$ if there exists $f(z,w) \in L[z,w]$ such that the field of meromorphic functions on X is isomorphic to

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Question: How do we know when X is defined over $\overline{\mathbb{Q}}$?

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Theorem (G.V. Belyĭ 1979)

A curve X over $\mathbb{C}$ *can be defined over* $\overline{\mathbb{Q}}$ *if and only if there exists a branched covering of compact connected Riemann surfaces* $\varphi : X \to \mathbb{P}^1$ *unramified (unbranched) above* $\mathbb{P}^1 \setminus \{0, 1, \infty\}$*.*

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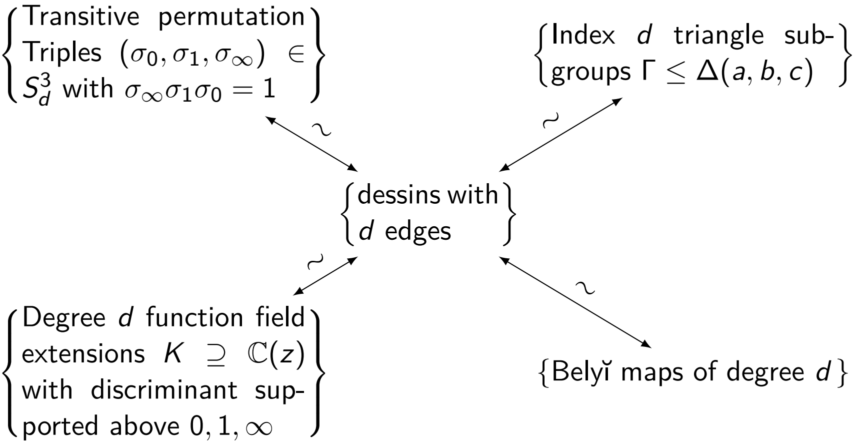
There is a zoo of objects in bijection with the set of Belyĭ maps.

A Zoo of Bijections



$\left\{\text{Transitive permutation Triples } (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3 \text{ with } \sigma_\infty\sigma_1\sigma_0 = 1\right\}$

$\left\{\text{Index } d \text{ triangle subgroups } \Gamma \leq \Delta(a, b, c)\right\}$

$\left\{\text{dessins with } d \text{ edges}\right\}$

$\left\{\text{Degree } d \text{ function field extensions } K \supseteq \mathbb{C}(z) \text{ with discriminant supported above } 0, 1, \infty\right\}$

$\{\text{Belyĭ maps of degree } d\}$

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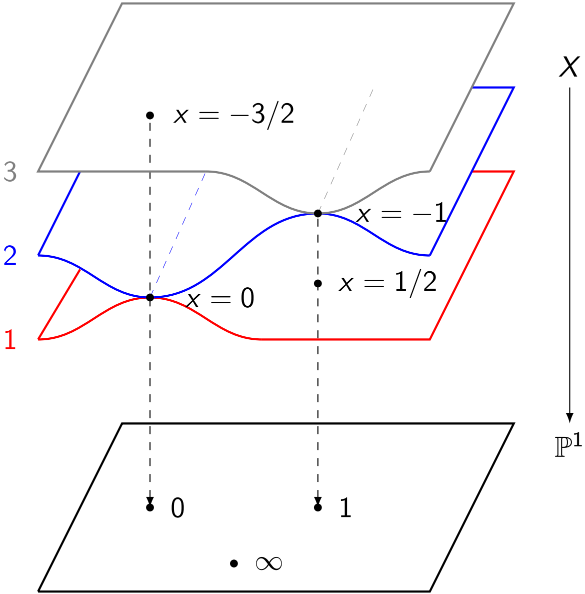

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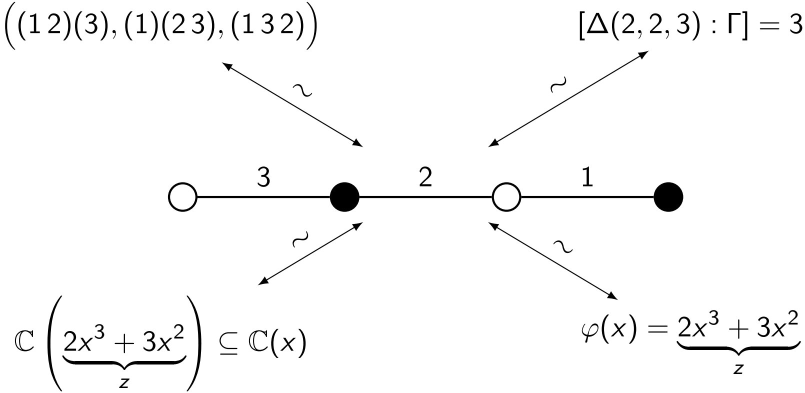

$$\big((1\,2)(3), (1)(2\,3), (1\,3\,2)\big)$$

$$[\Delta(2,2,3) : \Gamma] = 3$$

$$\mathbb{C}\left(\underbrace{2x^3+3x^2}_{z}\right) \subseteq \mathbb{C}(x)$$

$$\varphi(x) = \underbrace{2x^3+3x^2}_{z}$$

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This yields a Galois action on everything in the zoo.

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Also in Grothendieck's Esquisse d'un Programme, he writes about the computation of specific examples of Belyĭ maps:

Exactly which are the conjugates of a given oriented map? (Visibly, there is only a finite number of these.) I considered some concrete cases (for coverings of low degree) by various methods, J. Malgoire considered some others–I doubt there is a uniform method for solving the problem by computer.

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In 2014, KMSV provide a general purpose numerical method for computing Belyĭ maps using power series expansions of modular forms.

Database

We are currently tabulating a database of *all* Belyĭ maps in low degree to be included in the LMFDB `www.lmfdb.org`.

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Below is a table detailing how many Belyĭ maps (up to isomorphism) there are of given degree and genus.

d	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g > 3$	total
2	1	0	0	0	0	1
3	2	1	0	0	0	3
4	6	2	0	0	0	8
5	12	6	2	0	0	20
6	38	29	7	0	0	74
7	89	50	13	3	0	155
8	261	217	84	11	0	573
9	583	427	163	28	6	1207

Galois action for example
`7T5-[3,4,4]-331-421-421-g0`

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$$\varphi = \lambda \cdot \frac{x^3\left(x - \frac{1}{729}(68\sqrt{7} + 236)\right)^1 \left(x - \frac{1}{9}(20 - 4\sqrt{7})\right)^3}{\left(x - \frac{4}{21}(\sqrt{7} + 3)\right)^2 \left(x - \frac{4}{21}(\sqrt{7} + 1)\right)^4}$$

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$$\varphi - 1 = \lambda \cdot \frac{\left(x - \frac{1}{189}\left(44\sqrt{7} + 140\right)\right)^4 \left(x - \frac{1}{7}\left(12\sqrt{7} - 28\right)\right)^2 \left(x - \frac{1}{14}\left(3\sqrt{7} + 7\right)\right)^1}{\left(x - \frac{4}{21}\left(\sqrt{7} + 3\right)\right)^2 \left(x - \frac{4}{21}\left(\sqrt{7} + 1\right)\right)^4}$$

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$$\begin{aligned}\sigma_0 &= (1\,3\,7)(2)(4\,5\,6)\\ \sigma_1 &= (1\,4\,5\,3)(2\,7)(6)\\ \sigma_\infty &= (1\,2\,7\,5)(3)(4\,6)\end{aligned} \quad \xleftrightarrow{\qquad\tau\qquad} \quad \begin{aligned}\sigma_0 &= (1\,3\,7)(2)(4\,5\,6)\\ \sigma_1 &= (1\,6\,3\,2)(4\,5)(7)\\ \sigma_\infty &= (1\,7\,6\,4)(2\,3)(5)\end{aligned}$$

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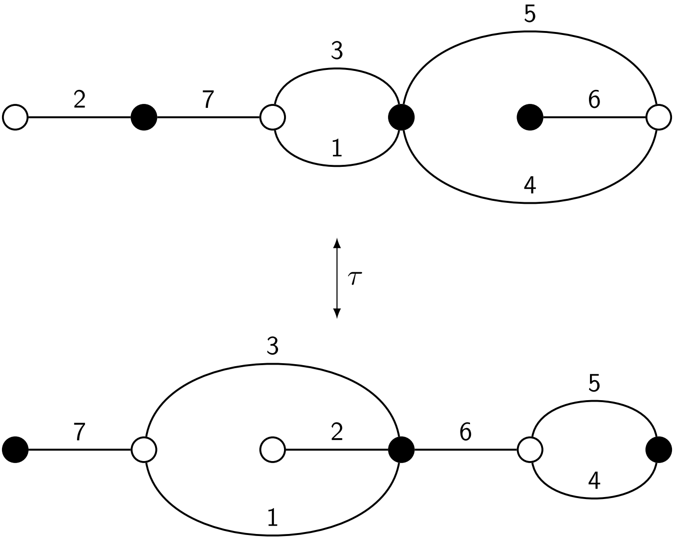

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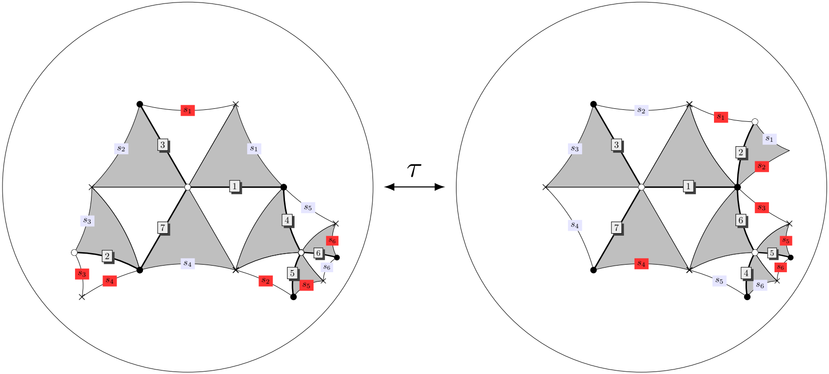

Thanks for listening!