# GROWTH RATE OF A LIQUIDITY PROVIDER'S WEALTH IN $XY = c$ AUTOMATED MARKET MAKERS

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Abstract. We study the geometric return of a discretized version uniswap automated market maker (AMM) under the no-arbitrage assumption. In particular, we give a formula for the growth rate of the liquidity provider (LP) wealth in the presence of fees.

## 1. introduction

We consider AMMs which follow the standard $XY = c$ curve popularized by the Uniswap platform and studied among others in [1, 2]. We aim to understand the expected growth rate of the wealth of a liquidity provider (LP) in the presence of fees. We will assume the existence of an external which has access to infinite liquidity and trade against the AMM every time there is an arbitrage opportunity. The key point for us will be to quantify the trade-off between the frequency of fee-collecting trades for the LP and the amount of fee collected at each trade. To achieve this goal we choose a discretization of the model which allows us to use standard results on stationary distribution from Markov chains theory. In particular we are able to derive in Proposition 3.1 a closed formula for the expected growth rate of the wealth.

## 2. The model

We denote by $X_t$ (resp. $Y_t$) the quantity of numeraire (resp. asset) held by the LP in the AMM at time $t$ and by $\gamma$ the percentage of money spent by a trader that goes into a trade. Informally $1-\gamma$ is the fee charged by the AMM. We fix $p \in [\frac{1}{2}, 1]$, $\{U_i\}_{i\in\mathbb{N}}$ to be a sequence of i.i.d random variables with $\mathbb{P}(U_1 = 1) = p$ and $P(U_1 = -1) = (1-p)$ and set $\delta > 0$. The variable $p$ encodes the drift of our price process while $\delta$ is a discrete price step which encodes how much the price of the asset can jump at each discrete time step.

**Definition 2.1** (Price process). *The price process of the asset is defined as*

$$S_t = \begin{cases} 1 & \text{if} \quad t = 0 \\ S_{t-1}e^{\delta U_t} & \text{if} \quad t > 1 \end{cases} \tag{1}$$

**Remark 2.2.** *The case $p < \frac{1}{2}$ can be derived from by inversing the role played by the asset and the numeraire*

**Definition 2.3** (Wealth process and implicit price). *For $t \geq 0$, denote by $X_t$ and $Y_t$ the quantity of numeraire and asset respectively provided by the LP in the AMM. The wealth of the LP at time $t$ is defined as:*

$$W_t = X_t + Y_t S_t$$

*The implicit price $S_t^*$ given by the curve is defined as $S_t^* = \frac{wX_t}{(1-w)Y_t}$ and represents the price at which a trader can start trading against the AMM in the absence of fee.*

Our goal is to find the expected geometric return of the wealth process as a function of $\gamma$. For reasons that will appear clearer later, we constrain the fee $\gamma$ to be of the form $e^{-k_\gamma \delta}$ for some integer $k_\gamma$.

**Definition 2.4** (AMM mechanism). *A trader at time $t$ can trade against the AMM an amount $\Delta^* X_t = (1+\gamma)\Delta X_t$ (resp. $\Delta^* Y_t$) of numeraire (resp. asset) for an amount $\Delta Y_t$ (resp. $\Delta X_t$) of asset (resp. numeraire) using the following relation:*

$$(X_t + \Delta X_t)^{w\gamma}(Y_t - \Delta Y_t)^{1-w} = X_t^{w\gamma} Y_t^{1-w}$$

*It can be shown that, in the case of a numeraire based trade for example, the starting implicit price paid by the trader is given by $\frac{\Delta X_t}{\Delta Y_t} = \frac{\gamma(1-w)}{w}\left(\frac{X_t}{Y_t}\right)$.*


(A) No trade happens at time $t$ because $S_t \leq \gamma^{-1} S_t^*$ (B) A trade happens at time $t$ because $S_t > \gamma^{-1} S_t^*$

A visualization of the Markov chain that dictates the ratio $\frac{S_t}{S_t^*}$

**Remark 2.5.** *The following observation will be useful to or calculation. If* $S_t = \frac{\gamma(1-w)}{w}\left(\frac{X_t}{Y_t}\right)e^{\delta}$, *the optimal trade resulting from the no-arbitrage condition is constrained by the system of equations:*

$$\left(\frac{X_t + \Delta X_t}{Y_t - \Delta Y_t}\right) = \left(\frac{X_t}{Y_t}\right) e^{\delta}$$
$$(X_t + \Delta X_t)^{w\gamma}(Y_t - \Delta Y_t)^{1-w} = X_t^{w\gamma} Y_t^{1-w}$$

*It is straightforward to check that this implies:*

$$\Delta X_t = X_t\left(e^{\frac{(1-w)\delta}{w\gamma+1-w}} - 1\right)$$
$$\Delta Y_t = Y_t\left(1 - e^{\frac{-w\gamma\delta}{w\gamma+1-w}}\right)$$

At time $t = 0$ both the implicit price on the curve and the price $S_0$ are equal to 1. We always suppose that the price $S_t$ before the trader decides or not to trade against the AMM. If at time $t$ the price $S_t$ is strictly greater (resp. strictly smaller) than the implicit price that the trader would get by trading an amount $\Delta X_t$ (resp. $\Delta^* Y_t$), the trader will execute the trade which maximizes his arbitrage. It is not hard to see that, due to our restrictions on the set of possible fees, the trader will always change the implicit price on the curve by either a factor $e^{\delta}$ or a factor $e^{-\delta}$. This allows to restate the problem in the following terms:

- At $t = 0$, we have $S_0 = X_0 = Y_0 = 1$.
- If at time $t$: $S_t = e^{(k_\gamma+1)\delta} S_t^*$, the trader will exchange a quantity $\Delta X_t = X_t(e^{\frac{\delta}{\gamma+1}} - 1)$ of numeraire for $\Delta Y_t = Y_t(1 - e^{\frac{-\gamma\delta}{\gamma+1}})$ of the asset. The new implicit price given by the curve after the exchange is $S^*_{t+\Delta t} = \gamma^{-1} S_t$.
- If at time $t$: $S_t = e^{-\delta}\gamma S_t^*$, the trader will exchange a quantity $\Delta Y_t = Y_t(e^{\frac{\gamma\delta}{\gamma+1}} - 1)$ of the asset for $\Delta X_t = X_t(1 - e^{\frac{-\gamma\delta}{\gamma+1}})$ of the numeraire. The new implicit price given by the curve after the exchange is $S^*_{t+\Delta t} = \gamma S_t$. The implicit price given by the AMM doe snot changes.
- If at time $t$: $e^{-k_\gamma\delta} S_t^* \leq S_t \leq e^{k_\gamma\delta} S_t^*$, the trader does not interact with the AMM since no amount of trading is profitable.

This means that $M_t = \log \frac{S_t}{S_t^*}$ is a Markov chain with state space $\{-k_\gamma\delta, (1-k_\gamma)\delta ...., k_\gamma\delta\}$. A trades occur everytime the Markov chain $M_t$ stays on one of the two end states.

$$\begin{pmatrix} 1-p & p & & & & & \\ 1-p & 0 & p & & & & \\ 0 & 1-p & 0 & p & & & \\ & & & \cdots & & & \\ & & & & \cdots & & \\ & & & & & \cdots & \\ & & & & 1-p & 0 & p \\ & & & & & 1-p & p \end{pmatrix}$$

It is not had to verify that this Markov chain has stationary distribution

$$\left[\frac{1}{2k_\gamma+1}, ..., \frac{1}{2k_\gamma+1}\right] \tag{2}$$

if $p=\frac{1}{2}$, and

$$\left[\frac{1-\frac{1-p}{p}}{\left(\frac{p}{1-p}\right)^{2k_\gamma+1}-1}, \frac{\left(1-\frac{1-p}{p}\right)\frac{p}{1-p}}{\left(\frac{p}{1-p}\right)^{2k_\gamma+1}-1}, \ldots, \frac{\left(1-\frac{1-p}{p}\right)\left(\frac{p}{1-p}\right)^{2k_\gamma+1}}{\left(\frac{p}{1-p}\right)^{2k_\gamma+1}-1}\right] \tag{3}$$

if $p\in(\frac{1}{2},1)$.

Another key property of this model is that, after each trade from cash to the asset the product $C_t = X_t^w Y_t^{1-w}$ is increased by the same factor $K_\gamma$ given by:

$$\begin{aligned}
K_\gamma^w &= \frac{(X_t+\Delta X_t)^w (Y_t-\Delta Y_t)^{1-w}}{X_t^w Y_t^{1-w}} && (4)\\
&= \left(\frac{X_t+\Delta X_t}{X_t}\right)^{w(1-\gamma)} \frac{(X_t+\Delta X_t)^{w\gamma}(Y_t-\Delta Y_t)^{1-w}}{X_t^{w\gamma}Y_t^{1-w}} && (5)\\
&= \left(1+\frac{\Delta X_t}{X_t}\right)^{w(1-\gamma)} && (6)\\
&= e^{\frac{\delta(1-\gamma)}{\frac{\gamma}{1-w}+\frac{1}{w}}} && (7)
\end{aligned}$$

## 3. Analysis of the expected geometric return

We are now ready to compute the asymptotic geometric return for the LPs.

**Proposition 3.1.** *For an interval of time $\Delta t = \frac{1}{n}$ the growth rate of the LP wealth is given by:*

$$\lim_{T\to\infty}\frac{\mathbb{E}[W_T]}{T} = \begin{cases} \frac{n\delta}{2}\frac{1-\gamma}{1-2\frac{\log\gamma}{\delta}}\left(\frac{1}{\frac{\gamma}{1-w}+\frac{1}{w}}+\frac{1}{\frac{\gamma}{w}+\frac{1}{1-w}}\right) & \text{if} \quad p=\frac{1}{2}\\ \frac{n\delta(2p-1)(1-\gamma)}{1-\left(\frac{1-p}{p}\right)\gamma^{\frac{2}{\delta}\log\left(\frac{p}{1-p}\right)}}\left(\frac{1}{\frac{\gamma}{1-w}+\frac{1}{w}}+\frac{\gamma^{\frac{2}{\delta}\log\left(\frac{p}{1-p}\right)}}{\frac{\gamma}{w}+\frac{1}{1-w}}\right)+n\delta(2p-1)(1-w) & \text{if} \quad p>\frac{1}{2}\end{cases}$$

*Proof.* Notice that since the no arbitrage condition imposes $\gamma \le \frac{S_t}{S_t^*} \le \gamma^{-1}$, we have the natural bound $\gamma \le \frac{Y_tS_t+X_t}{Y_tS_t^*+X_t} \le \gamma^{-1}$ for the ration between the real wealth and the implicit wealth.

This allows us to rewrite the expected log wealth as:

$$\begin{aligned}
\mathbb{E}[\log W_t] &= \mathbb{E}[\log(Y_tS_t+X_t)]\\
&= \mathbb{E}[\log(Y_tS_t^*+X_t)+\log\left(\frac{Y_tS_t+X_t}{Y_tS_t^*+X_t}\right)]\\
&= \mathbb{E}[\log(Y_tS_t^*+X_t)]+O(1)
\end{aligned}$$

If we set $M_t$ and $N_t$ to be the number of trades occuring until time $t$ included against the cash and asset respectively , $(K_\gamma^w)^{-M_t}(K_\gamma^{1-w})^{-N_t}(Y_tS_t^*+X_t)$ corresponds to the wealth in the AMM at time $t$ in the absence of fees and can be rewritten as

$$\begin{aligned}
(K_\gamma^w)^{-M_t}(K_\gamma^{1-w})^{-N_t}(Y_tS_t^*+X_t) &= X_t^{-w}Y_t^{w-1}(Y_tS_t^*+X_t)\\
&= X_t^{-w}Y_t^{w-1}\left(Y_t\frac{(1-w)X_t}{wY_t}+X_t\right)\\
&= \frac{(1-w)}{w}\left(\frac{X_t}{Y_t}\right)^{1-w}
\end{aligned}$$

We are now ready to compute the asymptotic geometric return of the LP wealth.

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\mathbb{E}[\log W_T] &= \lim_{T\to\infty}\frac{1}{T}\left(\mathbb{E}[\log(Y_T S_T^* + X_T)] + O(1)\right)\\ &= \lim_{T\to\infty}\frac{1}{T}\mathbb{E}[\log(K_\gamma^w)^{\frac{M_t}{2}}(K_\gamma^{1-w})^{\frac{N_t}{2}} + (1-w)\log S_T^*]\\ &= \lim_{T\to\infty}\frac{\log K_\gamma^w \mathbb{E}[M_T] + \log K_\gamma^{1-w}\mathbb{E}[N_T]}{2T} + \lim_{T\to\infty}\frac{1-w}{T}\mathbb{E}[\log S_T^*]\\ &= \lim_{T\to\infty}\frac{\log K_\gamma^w \mathbb{E}[M_T] + \log K_\gamma^{1-w}\mathbb{E}[N_T]}{2T} + n\delta(1-w)(2p-1)\end{aligned}$$

Hence we are only left to compute $\lim_{T\to\infty}\frac{\mathbb{E}[M_T]}{T}$ and $\lim_{T\to\infty}\frac{\mathbb{E}[N_T]}{T}$. This corresponds to the average number of trades during an interval of time while in the stationary distribution of the Markov chain. If we denote by $[\pi(-k_\gamma\delta), \pi\left((1-k_\gamma)\delta\right), ...., \pi(k_\gamma\delta)]$ the stationary distribution of the Markov chain described above. The average number of trades in an interval of time $T$ is given by $nT(1-p)\pi(-k_\gamma\delta)$ and $nTp\pi(k_\gamma\delta)$ for trade of cash versus asset and asset versus cash respectively . Using the expression given in equations (2) and (3), we obtain:

$$\lim_{T\to\infty}\frac{\mathbb{E}[M_T]}{nT} = \begin{cases}\frac{1}{2k_\alpha+1} & \text{if} \quad p=\frac{1}{2}\\ \frac{2p-1}{1-\left(\frac{1-p}{p}\right)^{2k_\alpha+1}} & \text{if} \quad p>\frac{1}{2}\end{cases}$$

$$\lim_{T\to\infty}\frac{\mathbb{E}[N_T]}{nT} = \begin{cases}\frac{1}{2k_\alpha+1} & \text{if} \quad p=\frac{1}{2}\\ \left(\frac{1-p}{p}\right)\left(\frac{1-p}{p}\right)^{2k_\alpha}\frac{2p-1}{1-\left(\frac{1-p}{p}\right)^{2k_\alpha+1}} & \text{if} \quad p>\frac{1}{2}\end{cases}$$

We can rewrite these expressions using:

$$\frac{1}{2k_\alpha+1} = \frac{1}{1-2\frac{\log\gamma}{\delta}}$$

and $\left(\frac{1-p}{p}\right)^{2k_\alpha} = \gamma^{\frac{2}{\delta}\log\left(\frac{p}{1-p}\right)}$.

Now we simply have to replace $K_\gamma^w$ and $K_\gamma^{1-w}$ by their expression computed in (4) to finish our proof. □

## 4. Convergence to asset prices following a Geometric Brownian Motion

**$p = \frac{1}{2}$.** If we set $\Delta t = \frac{1}{n}$ and $\delta_n = \frac{\sigma}{n^{\frac{1}{2}}}$, the process $\log S_t$ converges to a Brownian motion with variance $\sigma$ as $n$ goes to $\infty$. The corresponding limit for the growth rate of the wealth of an $LP$ is given by:

$$-\frac{\sigma^2}{4\log\gamma}(1-\gamma)\left(\frac{1}{\frac{\gamma}{1-w}+\frac{1}{w}} + \frac{1}{\frac{\gamma}{w}+\frac{1}{1-w}}\right) \tag{8}$$

**$p > \frac{1}{2}$.** The parameters that makes the price process converges to a geometric Brownian motion satisfy the system of equations:

$$\delta_n\sqrt{p_n(1-p_n)} = \frac{\sigma}{2n^{\frac{1}{2}}}$$
$$\delta_n(2p_n-1) = \left(\mu-\frac{\sigma^2}{2}\right)\frac{1}{n}$$

If we set $d = \mu - \frac{\sigma^2}{2}$ and $\alpha_n = \frac{2d}{\sigma n^{\frac{1}{2}}}$, this imposes $p_n = \frac{1}{2} + \frac{\alpha_n}{2\sqrt{\alpha_n^2+4}}$. In particular $p_n = \frac{1}{2} + \frac{d}{2\sigma n^{\frac{1}{2}}} + O(n^{-\frac{1}{2}})$ and $\delta_n = \frac{\sigma}{n^{\frac{1}{2}}} + O(n^{-\frac{1}{2}})$.

Plugging those values in the formula Proposition 3.1 gives us an asymptotic growth rate of:

$$d\left(\frac{1-\gamma}{1-\gamma^{\frac{4d}{\sigma^2}}}\left(\frac{1}{\frac{\gamma}{1-w}+\frac{1}{w}}+\frac{\gamma^{\frac{4d}{\sigma^2}}}{\frac{\gamma}{w}+\frac{1}{1-w}}\right)+(1-w)\right) \tag{9}$$

For fixed $\mu$,$\sigma$ and $w$ the optimal growth rate is given by:

$$\max\left(d, \frac{\sigma^2}{2}w(1-w)+d(1-w)\right)$$

In particular for $\frac{\sigma^2}{2} \le \mu \le \sigma^2$ , the optimal weight is given by $w = 1-\frac{\mu}{\sigma^2}$ for a corresponding growth return of $\frac{\mu^2}{2\sigma^2}$.

$p < \frac{1}{2}$**.** In this case we can reverse the role of the numeraire and the asset from the case $p > \frac{1}{2}$ and change back to express the wealth in terms of numeraire at the end. It turns out that it leaves the expression above invariant.

## 5. Optimal fees and optimal growth

**Zero drift (**$p = \frac{1}{2}$ **or equivalently** $d = 0$**).** We observe that the function $x \to \frac{1-x}{1+x}$ is increasing on $(0,1)$ and so is the function $x \to -\frac{1}{\log x}\frac{1-x}{1+x}$. As a consequence $\gamma$ should be as close as possible to 1 without being 1 and the optimal wealth growth is given by:

$$\lim_{\gamma\to 1} -\frac{\sigma^2}{4\log\gamma}\frac{1-\gamma}{\gamma+1} = \frac{\sigma^2}{8} \tag{10}$$

Notice also that in this case the growth rate of the unrebalanced portfolio is zero (see appendix) and the growth rate of an LP period is always positive. This means that the growth rate of an LP position is outperforming the unrebalanced portfolio for any $\gamma$.

**Positive drift (**$p > \frac{1}{2}$ **or equivalently** $d > 0$**).** The function $x \to \frac{1+x^a}{1-x^a}\frac{1-x}{1+x}$ is increasing on $(0,1)$ for $a > 1$, decreasing for $a < 1$ and constant for $a = 1$. This tells us that, in order to maximize the growth rate:

- If $\frac{4d}{\sigma^2} > 1$ or equivalently $\mu < \frac{3\sigma^2}{2}$, $\gamma$ should be as large as possible and the best possible growth rate is:

$$\lim_{\gamma\to 1}\frac{d}{2}\left(\left(\frac{1+\gamma^{\frac{4d}{\sigma^2}}}{1-\gamma^{\frac{4d}{\sigma^2}}}\right)\left(\frac{1-\gamma}{1+\gamma}\right)+1\right) = \frac{d}{2}+\frac{\sigma^2}{8} \tag{11}$$

- If $\frac{4d}{\sigma^2} < 1$ or equivalently $\mu > \frac{3\sigma^2}{2}$, $\gamma$ should be set to 0 (no trade) and the best possible growth rate is:

$$\lim_{\gamma\to 0}\frac{d}{2}\left(\left(\frac{1+\gamma^{\frac{4d}{\sigma^2}}}{1-\gamma^{\frac{4d}{\sigma^2}}}\right)\left(\frac{1-\gamma}{1+\gamma}\right)+1\right) = d \tag{12}$$

- If $\mu = \frac{3\sigma^2}{2}$, the growth rate is constantly equal to $d$ independently of $\gamma$ .

Moreover, as shown in the appendix, the growth rate of an unrebalanced portfolio is $d$. This implies that the growth rate of an LP is always superior to the unrebalanced portfolio if $\frac{4d}{\sigma^2} < 1$ and always inferior to the unrebalanced portfolio if $\frac{4d}{\sigma^2} > 1$.

**Negative drift drift (**$p < \frac{1}{2}$ **or equivalently** $d < 0$**).** In this case, the equation 9 is still valid and the conclusions are the same as for the positive drift with $\frac{4d}{\sigma^2} > 1$. In particular, the optimal fee is $\gamma$ as close as possible to 1, the optimal growth rate is $\frac{d}{2}+\frac{\sigma^2}{8}$ and the LP always outperform an unrebalanced portfolio for $\gamma$.

## 6. Appendix

**Lemma 6.1.** *For $a, b > 0$ and and a price process $\{S_t\}$ with positive drift :*

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\log(aS_T + b)] = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\log S_T]$$

*Proof.* The logarithm function is $\ell$-Lipschitz for some constant $\ell$ on $[\frac{b}{a}, +\infty]$ thus

$$log(S_T + \frac{b}{a}) \leq \log S_T + \ell\frac{b}{a}$$

$$\log a + \log S_T = \log aS_T \leq \log(aS_T + b) = \log a + \log\left(S_T + \frac{b}{a}\right) \leq \log a + \ell\frac{b}{a} + \log S_T$$

and

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\log a + \log S_T] = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\log S_T] \leq \lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\log(aS_T + b)] \leq \lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\log a + \ell\frac{b}{a} + \log S_T] = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\log S_T]$$

This implies

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\log(aS_T + b)] = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\log S_T].$$

□

$$\begin{aligned} P(e^{rt}(bX)^{1/2} - \frac{X+b}{2}) \geq 0 =& P(4e^{2rt}bX \geq (X+b)^2) \\ =& P(X^2 + (2 - 4e^{2rt})bX + b^2 \geq 0) \\ =& P\left(X \leq b\left(2e^{2rt} - 1 + \left((2e^{2rt} - 1)^2 - 1\right)^{1/2}\right)\right) - P\left(X \leq b\left(2e^{2rt} - 1 - \left((2e^{2rt} - 1)^2 - 1\right)^{1/2}\right)\right) \end{aligned}$$

For a $lognormal(\mu, \sigma^2)$ this gives a probability

$$\Phi\left(\frac{\log\left(b\left(2e^{2rt} - 1 + \left((2e^{2rt} - 1)^2 - 1\right)^{1/2}\right)\right) - \mu}{\sigma}\right) - \Phi\left(\frac{\log\left(b\left(2e^{2rt} - 1 - \left((2e^{2rt} - 1)^2 - 1\right)^{1/2}\right)\right) - \mu}{\sigma}\right)$$

## References


[1] G. Angeris, H.-T. Kao, R. Chiang, C. Noyes, and T. Chitra. An analysis of uniswap markets. *Cryptoeconomic Systems Journal*, 2019.

[2] Y. Zhang, X. Chen, and D. Park. Formal specification of constant product (x× y= k) market maker model and implementation. *URL: ht tp s://gi th ub. c om/r un ti me ve ri fi ca ti on/v er if ie ds ma rt-c on tr ac ts/b lo b/un is wa p/un is wa p/xy-k. p df, Accessed: December*, 30:2019, 2018.