Limit shape “Aztec Diamond” for domino tilings

Colors represent the parity of the dominos
Overview

- We study random discrete maps.
- Limit shapes are a universal phenomenon.
- A variational principle explains:
  - the occurrence of limit shapes
  - the shape of the limit.
Overview

• Methods for deducing variational principles:
  • so far: based on integrability/lattice structure of the model.
  • now: based on Kirszbraun theorem and concentration inequality.

• Main result:
  Variational principle for graph homomorphisms to a tree

• First variational principle for a model of random discrete maps that is not integrable.
Graph homomorphism $h : G_1 \to G_2$:

$x \sim y \Rightarrow h(x) \sim h(y)$.

Example 1: Graph homomorphism to $\mathbb{Z}$:

$h : \mathbb{Z}^2 \supset \Lambda \to \mathbb{Z}$  "height function"

Example 2: Graph homomorphism to 3 regular tree $\mathcal{T}$:

$h : \mathbb{Z}^2 \supset \Lambda \to \mathcal{T}$  "Tree-valued height function"
Objects we study

Random graph homomorphism:

1. Fix $\Lambda \subset \mathbb{Z}^d$
2. Fix boundary data $g : \partial \Lambda \rightarrow \mathbb{Z}$
3. Pick uniformly at random one element of

$$\{ h : \Lambda \rightarrow \mathbb{Z} \mid h \text{ is height function and } h|_{\partial \Lambda} = g \}$$

For large $\Lambda$ and under proper rescaling you will see "limit shapes".
Limit shape for graph homomorphism into $\mathbb{Z}$

Colors represent the parity of the height in $\mathbb{Z}$
Limit shape for graph homomorphism into 3 regular tree $\mathcal{T}$
Appearance of limit shapes is a universal phenomenon
Limit shape for lozenge tilings

Each color represents a type of lozenge
Limit shape “Aztec Diamond” for domino tilings

Colors represent the parity of the dominos
Limit shape for tilings by 3-1 bars
Limit shape for ribbon tilings
Variational principle for domino tilings and height functions $h : \Lambda \rightarrow \mathbb{Z}$
Two questions:

- How many height functions $h : \Lambda \to \mathbb{Z}$ with given boundary values exist?
- What do they asymptotically look like?
Microscopic entropy

\[
\text{Ent}(\Lambda, g_{\partial\Lambda}) := \frac{1}{|\Lambda|} \ln \left| \left\{ h : \Lambda \rightarrow \mathbb{Z} \mid h \text{ is height function and } h_{|\partial\Lambda} = g \right\} \right|
\]

Microscopic surface tension

\[
\text{ent}_n(s, t) := \frac{1}{n^2} \ln \left| \left\{ h : B_n \rightarrow \mathbb{Z} \mid h \text{ is height function and for all } (x_1, x_2) \in \partial B_n : h(x_1, x_2) \approx s \cdot x_1 + t \cdot x_2 \right\} \right|
\]

with \( B_n = \{1, \ldots, n\}^2 \) and \(-1 \leq s, t \leq 1\)
Local surface tension

\[
\text{ent}(s, t) := \lim_{n \to \infty} \text{ent}_n(s, t)
\]

Macroscopic entropy

\[
\text{Ent}(R, f) := \int_R \text{ent}(\nabla f(x)) \, dx
\]

with \( R \subset \mathbb{R}^2 \) and \( f : R \to \mathbb{R} \) 1-Lipschitz
The variational principle for domino tilings

Theorem 1 (Cohn, Kenyon, Propp ‘00)

Assume: \( \frac{1}{n} \Lambda_n \to \mathbb{R} \) and \( g_{\partial \Lambda_n} \to g_{\partial \mathbb{R}} \)

Then: \( \operatorname{Ent}(\Lambda_n, g_{\partial \Lambda_n}) \to \sup_{f: f_{\partial \mathbb{R}} = g_{\partial g}} \operatorname{Ent}(\mathbb{R}, f) \).

Asymptotically characterizes the number of height-functions.
Additionally

\[ \text{ent}(s, t) \quad \text{is strict concave} \]

and therefore

\[
\sup_{f: f\partial R = g\partial g} \text{Ent}(R, f) = \max_{f: f\partial R = g\partial g} \text{Ent}(R, f) = \int_R \text{ent}(\nabla f_{\text{max}}(x)) \, dx.
\]

WHY DO LIMIT SHAPES APPEAR?
Given $f : \mathbb{R} \to \mathbb{R}$ and $\varepsilon > 0$ define

$$B(\varepsilon, f, \Lambda_n) := \{\text{height functions } h : \Lambda_n \to \mathbb{Z} \text{ that are } \varepsilon\text{-close to } f\}$$

**Theorem 2 (Cohn, Kenyon, Propp ’00)**

$$\frac{1}{|\Lambda_n|} \ln |B(\varepsilon, f, \Lambda_n)| \to \text{Ent}(R, f) + \text{Error}(\varepsilon)$$
Combination of both theorems yields

\[
\frac{1}{|\Lambda_n|} \ln |B(\varepsilon, f_{\text{max}}, \Lambda_n)| \approx \text{Ent}(f_{\text{max}})
\]

\[
\approx \text{Ent}_n(\Lambda_n, g_{\partial \Lambda_n}) = \frac{1}{|\Lambda_n|} \ln |Z(\Lambda_n, g_{\partial \Lambda_n})|
\]

This means:
Uniform measure on \(Z(\Lambda_n, g_{\partial \Lambda_n})\) concentrates around \(B(\varepsilon, f_{\text{max}}, \Lambda_n)\)
Main ingredients of the proof

First ingredient: \( \lim_{n \to \infty} \text{ent}_n(s, t) \) exists

Second ingredient: \( \lim_{n \to \infty} \text{ent}_n(s, t) = \lim_{n \to \infty} \text{ent}_{n,\text{free}}(s, t) \)

\[
\text{ent}_{n,\text{free}}(s, t) = \frac{1}{n^2} \ln \left| \{ h : B_n \to \mathbb{Z} \mid h \text{ is height function and for all} \right. \\
\left. (x_1, x_2) \in \partial B_n : \ |h(x_1, x_2) - s \cdot x_1 + t \cdot x_2| \lesssim \sqrt{n} \} \right|
\]
Cohn, Kenyon, Propp use direct computations provided by the dimer model and monotonicity of the model.

**PROOF WITHOUT USING INTEGRABILITY?**

Content of recent work with Georg Menz on ArXiv.

Derive variational principle for graph homomorphisms $h : \Lambda_n \to T$. 
Comparing both methods

Method based on integrability: not robust but very precise

- local surface tension \( \text{ent}(s, t) \) is explicitly known
- local surface tension \( \text{ent}(s, t) \) is strictly concave
- limit shapes \( f_{\text{max}} \) can be analyzed
- fluctuations can be analyzed
Comparing both methods

The new method: more robust but not precise

- existence of limit \( \lim_{n \to \infty} \text{ent}_n(s, t) \)
- local surface tension \( \text{ent}(s, t) \) not explicitly known
- local surface tension \( \text{ent}(s, t) \) is concave, strict concavity open problem
- applies to many other graphs.
How is integrability substituted?

Two ingredients:

- A Kirszbraun theorem for graphs: Describe when one can glue two graph homomorphisms together.

- Concentration inequality

\[
P \left( |h(x_1, x_2) - (sx_1 + tx_2)| \geq \varepsilon n \right) \lesssim \exp \left( -C\varepsilon^2 n \right)
\]
How are those ingredients used?

Showing \( \lim_{n \to \infty} \text{ent}_n(t, s) = \lim_{n \to \infty} \text{ent}_{n,\text{free}}(t, s) \). In other words microscopic fluctuation on the boundary do not modify the entropy.
We say that a bipartite graph $H$ satisfy the Kirszbraun property if any contracting map from a subset of $\mathbb{Z}^d$ to $H$ which have the right parity condition, can be extended to a graph homomorphism from the whole lattice $\mathbb{Z}^d$ to $H$.

This theorem assure that one can attach graph homomorphisms with similar boundary conditions without losing significant entropy.
Which graphs are Kirszbraun

Which graphs have the Kirszbraun property:

- All trees
- $\mathbb{Z}^2$
- Modified $\mathbb{Z}^3$ with extra points in the center of unit cubes

However for $d \geq 3$, the lattice $\mathbb{Z}^d$ is not Kirszbraun

In fact there exist a complete geometric characterization of Kirszbraun graphs based on triangles and geodesics between pairs of point (work in progress with Igor Pak and Nishant Chandgotia).
How is the concentration inequality deduced

For graph homomorphisms $h : B_n \to \mathbb{Z}$ several methods:

- monotonicity
- Random surfaces: cluster swapping

For graph homomorphisms $h : B_n \to T$ very difficult:

- Under the usual dynamic on $T$ not nice: configurations tends to diverge form each other

New coupling technique:

- based on Azuma-Hoeffding inequality
- uses a very carefully adapted Glauber dynamics
Difference with one-dimensional models

In a tree, you are not limited to climb on a single geodesic and limiting boundary conditions must be carefully redefined. Here is an illustration of the proper definition for the rescaling.

(a) $n=1$

(b) $n=2$

(c) $n=4$
The variational principle for tree homomorphisms

Theorem 3 (Menz, T.)

Assume: \( \frac{1}{n} \Lambda_n \rightarrow R \) and \( g_{\partial \Lambda_n} \rightarrow g_{\partial R} \)

Then: \( \text{Ent}(\Lambda_n, g_{\partial \Lambda_n}) \rightarrow \sup\limits_{f \text{ admissible}} \text{Ent}(R, f) \).

Here the sup is taken over a different set of functions due to the constraints imposed by the geodesics.
Differences with one-dimensional models

A limit shape for boundary conditions climbing on several geodesics
Open questions:

- Is the local entropy strictly concave?
- Is it possible to approximate the local entropy of a given slope in an efficient way?

Other models:

- Use different graphs instead of $T$ connected to tilings, random graphs.
- Use different lattices instead of $\mathbb{Z}^d$
- Consider weighted graphs instead of $T$ do homogenization.


Thank you!