Math 38 — Graph Theory

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Matrices for graph and Isomorphisms

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We saw last class that two graphs are the same if they are differently, as long as we are simply "moving the vertices". The goal of today's lecture is to make this statement more formal. One tool we will use is adjacency and incidence matrices. We will as well start classifying the graphs.

Matrices: adjacency matrix and incidence matrix

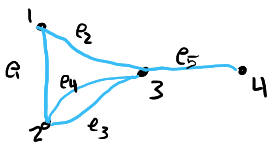
Let $G=(V, E)$ be a graph without any loop (it does not have to be a simple graph). We say that G, with vertices numbered 1 to n and edges numbered e_i for i in $\{1, 2, \ldots, m\}$.

The adjacency matrix of G, written $A(G)$, is the matrix whose entry a_{ij} is the number of edges with endpoints the vertices i and j.

The incidence matrix of G, written $M(G)$, is the n-by-m matrix whose (i,j)-entry is 1 if vertex i is an endpoint of edge j, and otherwise 0.

The incidence matrix is always a symmetric matrix.

The graph on the left has the following adjacency and incidence matrices:



$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad M(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The degree of a vertex (in a loopless graph) is the number of edges incident to that vertex.

Isomorphisms

So when are two graphs the same? We will answer this question using the notion of a bijection. As a reminder, this an injective and surjective function, or a one-to-one correspondence.

An isomorphism from a simple graph G to a simple graph H is a bijection $f: V(G) \to V(H)$ such that every edge uv of G is mapped to the edge $f(u)f(v)$ of H. We then say G and H are isomorphic, denoted $G \cong H$.

This is equivalent to asking that there exists a simultaneous permutation of the rows and columns of the adjacency matrix of G that would yield the adjacency matrix of H.

Example

The following graphs are isomorphic:

This is easily seen with the bijection that exchanges 1 and 3.

Remarks:

- Finding a bijection of the labels is the way to prove two graphs are isomorphic. However, to prove they are not isomorphic, there are many ways. For example, if the list of degrees is not the same, you will never be able to find an isomorphism. Or if the number of edges do not correspond. Among others.
- The isomorphism relation is an equivalence relation, i.e. this is a symmetric relation ($G \cong H$ iff $H \cong G$), a transitive relation ($G \cong H$ and $H \cong J$ imply $G \cong J$) and a reflexive one ($G \cong G$). That means that we can split the graph into equivalence classes.

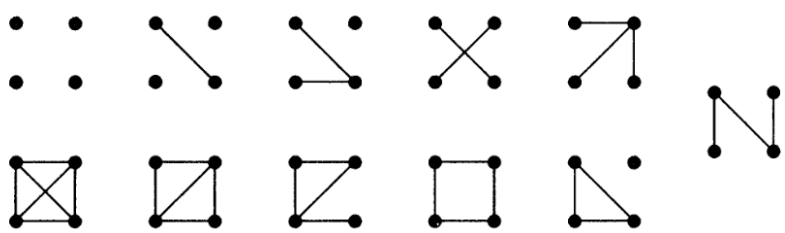
Example

The following graphs are not isomorphic. They both have six vertices, all of degree 3, and nine edges, and they are both connected, but one is bipartite and the other is not. Since they don't have the same properties, they are not isomorphic.

No triangle appear in the first graph.

Example

All the isomorphism classes for graphs with 4 vertices are



Special graphs

There are some graphs that have special names, and that turns out to be handy for whenever we want to use them or to classify them.

Complete graphs: Graphs with n vertices and $\binom{n}{2}$ edges.

K_n

K_5

Complete bipartite graphs: Bipartite graphs with independent sets of size s and r, with sr edges.

$K_{s,r}$

$K_{4,2}$

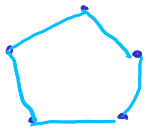
Paths: Connected graphs, with all the vertices of degree 2, except at most two who have degree 1.

P_n

P_4

Cycles: Paths with as many edges as vertices.

C_n



C_5

The complement of the graph G is the graph that has the same vertices and whose edges are all the edges that do not belong to G:

$K_{|G|} - E(G) = \overline{G}$

A graph G is self-complementary if its complement $\overline{G}$ is isomorphic to G.

Example: C_5 is self-complementary.

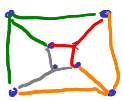
$K_5 =$   $\cong$ 

A decomposition of a graph is a list of subgraphs in which every edge appears exactly once.

Proposition

A graph G is self-complementary if and only if the complete graph is a decomposition into two copies of G.

Example: The cube decomposed into copies of $K_{1,3}$

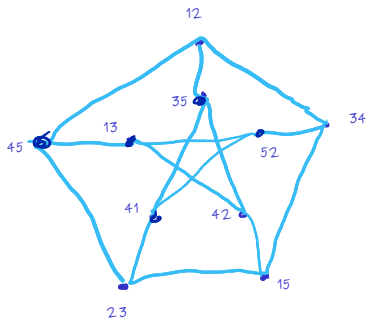


Note: $K_{1,3}$ is often called the claw.

The Petersen graph

The Petersen graph is a 10-vertices graph with 15 edges that is very famous, as it is an example or a counter-example to many phenomena.

The Petersen graph is the graph of 2-element subsets of $\{1,2,3,4,5\}$, and there is an edge between 2 subsets if their intersection is empty.



Some properties of the Petersen graph:

- Two non-adjacent vertices share exactly one neighbor.
- The graph has no triangle, but is not bipartite.
- The shortest cycle in the Petersen graph has length 5. (The length of the shortest cycle in a graph is called the girth of the graph.)

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 1.1