Last class, we considered connectivity to be the minimum number of vertices one can remove to get a disconnected graph. Today, we will look at a similar notion, but this time we are removing edges.
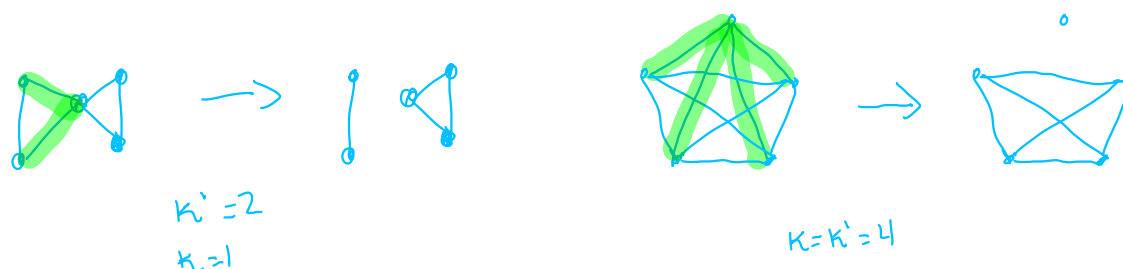
Definition

A <u>disconnecting set</u> is a subset of edges $F \subseteq E$ such that $G-F$ has at least 2 components.

separating ≠ disconnecting

The edge-connectivity is the minimum size of a disconnecting set, and is noted $\kappa'(G)$. A graph is <u>k-edge-connected</u> if it has edge-connectivity at least k.
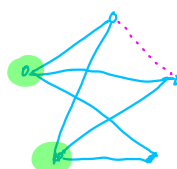
Examples

$\kappa' = 2$
$\kappa = 1$

$\kappa = \kappa' = 4$

Complete graphs have edge-connectivity n−1. You can prove it!

Let $S \subseteq V$ be a vertex subset of a connected graph G. Let $[S, \bar{S}]$ be the set of all edges with one endpoint in S and one in $\bar{S}$. Then $[S, \bar{S}]$ is an <u>edge cut</u>.
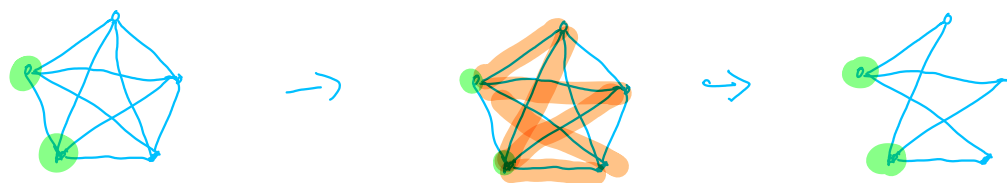
Edge cut $\underset{\not\Leftarrow}{\Rightarrow}$ Disconnecting set

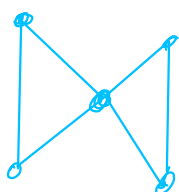
Edge cut ⇔ Minimal disconnecting set
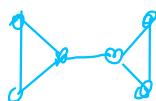
Theorem (Whitney, 1932)
If G is simple, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. In words: vertex-connectivity is at most edge-connectivity, which is always at most the smallest degree.
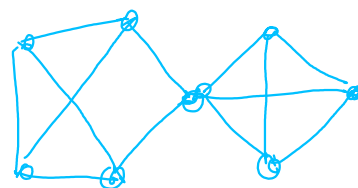
Example of inequalities

$\kappa(G) < \kappa'(G) = \delta(G)$ $\kappa(G) = \kappa'(G) < \delta(G)$ $\kappa(G) < \kappa'(G) < \delta(G)$

Proof
We first prove $\kappa'(G) \leq \delta(G)$. Let v be a vertex with degree $\delta(G)$. The edge cut for the set $\{v\}$ has $\delta(G)$ edges, so an edge cut with $\delta(G)$ edges exist, and the minimum edge cut has size at most $\delta(G)$.

We also need to prove $\kappa(G) \leq \kappa'(G)$. To do so, we start with a minimum edge cut, and construct a vertex cut with at most the same size. If this process is always possible, that proves the desired inequality.

Consider a minimum edge cut $[S, V-S]$. There are two cases:
- If every vertex of S is connected to every vertex of $V-S$, then $\#[S,V-S]=|S||V-S| \geq |V|-1$. Also, we proved last class that $\kappa(G) \leq |V|-1$. So $\kappa(G) \leq |V|-1 \leq \#[S,V-S]=\kappa'(G)$.
- Otherwise, there is one vertex x in S and y not in S that are not adjacent. We construct a set of vertices T:

    - All neighbors of x in $V-S$.
    - All vertices of $S\setminus\{x\}$ that are adjacent to vertices in $V-S$.

Then, T is a vertex cut: There is no way to go from x to y without passing through one edge of T, so G−T is disconnected. We need to show that T has at most #[S, V−S] vertices.

For each vertex t of T:

− If t is a neighbor of x, then xt is in the edge cut.

− If t is in S, then t is adjacent to at least one vertex u in V−S. Then ut is in the edge cut.
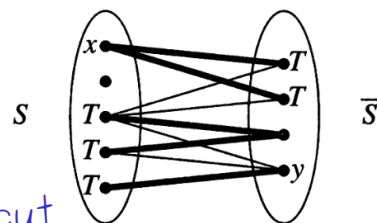
No edge is counted twice in this list, because x is not in T. Since every edge in this list is in the edge cut, then $|T| \leq$ #[S, V−S], and $\kappa(G) \leq \kappa'(G)$.

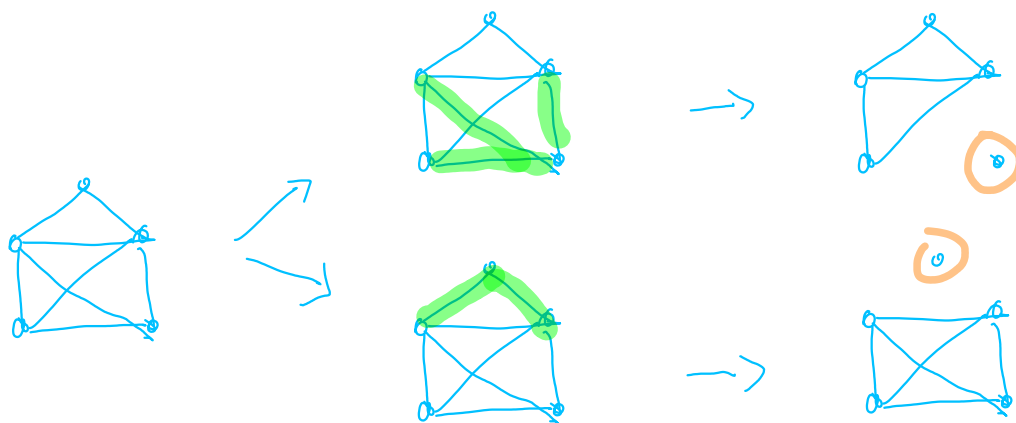## Proposition
Let G be a connected graph. Then, an edge cut F is minimal if and only if G−F has exactly two components.

Remark
If we replace minimal by minimum, then the statement becomes false: G−F can have two components while there are edge cuts with size smaller than |F|.

With your study group, try to agree on an explanation of why this is true.

Edge connectivity for regular graphs

Theorem
If G is a 3-regular graph, then $\kappa(G) = \kappa'(G)$.

Proof
We already know that $\kappa(G) \leq \kappa'(G)$, in general. To prove the statement, we only need to show the reverse inequality ($\geq$), that is, from a minimum vertex cut, create an edge cut of the same size.
Let S be a minimum vertex cut, and let H and J be two components of G−S. Since S is minimum, every vertex of it has a neighbor in H and a neighbor in J. Also a vertex cannot have at least two neighbors in both H and J since G is 3-regular. For each vertex v in S, delete the edge from v to the component in which in has only one neighbor (if there is one neighbor in H, one in J and another one (in S for example), delete the edge to H).
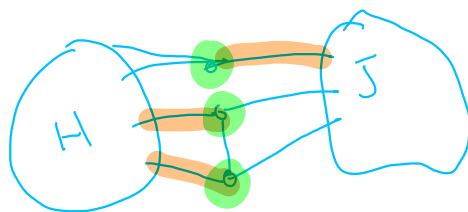
That process breaks all the paths between H and J, so the deleted edges form an edge cut. Also, the size of that edge cut is |S|, which proves the statement.

Are there any other values of r such that any r-regular simple graph G inevitably has $\kappa(G) = \kappa'(G)$? Discuss it with your study group.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 4.1