We keep looking at the interconnections between edge-connectivity and vertex-connectivity. We also consider what it means for cycles and paths.

Blocks

Is a connected graph with no cut-vertex 2-connected?

Connectivity 0   Connectivity 1

Definition

A <u>block</u> of a graph G is a maximal connected subgraph that has no cut-vertex.


Properties

- Isolated vertices, as well as "isolated edges" (isolated copies of K_2) are blocks.
- A cycle is always 2-connected, so it is always inside the same block.
- Since the only edges that are not in cycles are cut-edges, an edge with its two enpoints is a block if and only if it is a cut-edge.
- Blocks in a tree are edges, along with its two endpoints.
- Blocks in a loopless graph are its isolated vertices, its cut-edges and its 2-connected components.

Proposition

Two blocks in a graph share at most one vertex.

Proof

By contradiction. If two blocks A and B share vertices u and v, they are connected components with no cut-vertices inside. However, since these components are maximal, adding any edge would create a cut-vertex.
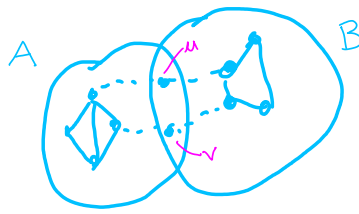
Since there is a path from u to v in A and one in B (because blocks are connected), there is a cycle containing u and v and A and B form together a 2-connected component, and hence are in the same block.

Proposition
If two blocks share a vertex, it is a cut-vertex.

2-connected graphs
Two paths from u to v are _internally disjoint_ if they have no common internal vertex.

Theorem (Whitney, 1932)
A graph with at least three vertices is 2-connected if and only if there exist internally disjoint u,v-paths for each pair {u,v}.

Proof
<= Since there are at least 2 disjoint u,v-paths for every pair {u,v}, u and v cannot be separated by removing one vertex. This is true for all {u,v}, so the graph does not have connectivity 1. It must have connectivity at least 2, and is hence 2-connected.

=> By induction on d(u,v).
  Base case: u and v are adjacent. Since the graph is 2-connected, it is also 2-edge-connected, and removing edge e={u,v} lets the graph connected, which means there is a path between u and v avoiding e.
 Induction hypothesis: If distance is k=d(u,v), there exists two internally disjoint uv-paths.
  Induction step: Let u and v be at distance k+1, and let P be a uv-path of (minimal) length k+1. Let w be the vertex on P at distance k of u, so w is adjacent to v, and P' be that portion of P.

By induction hypothesis, there exist two internally disjoint uw-paths, P' and Q'. If Q' contains vertex v, let Q be the portion from u to v of Q'; then Q is a uv-path that is internally disjoint from P.

P: u w v   or   P: u w v
Q'   Q'
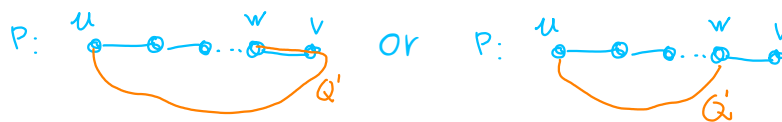
Otherwise, consider G-w. It is connected since there is no cut-vertex. So there is a path R between u and v avoiding w. If it avoids P or Q, R is internally disjoint from it. Otherwise, let x be the last vertex of R that also belongs to either P or Q. If x belongs to P, then Q is disjoint from the part of P between u and x and from the part of R between x and v, which is a path from u to v. If x belongs to Q, the argument is similar.
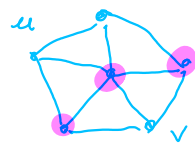
R
P: u w v
Q
x
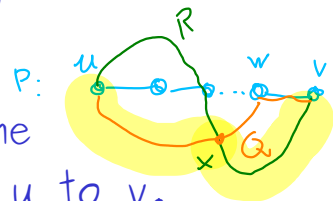
Corollary
For a graph with at least three vertices, the following conditions are characterization of 2-connected graphs:
(A) G is connected and has no cut-vertex.
(B) For every pair of vertices {u,v}, there are internally disjoint u,v-paths.
(C) For every pair of vertices {u,v}, there is a cycle through u and v.

Menger's theorem
Given two vertices u and v, a u̲v̲-̲c̲u̲t̲ is a set of vertices S such that G-S has no uv-path.
Let $\kappa(u,v)$ be the size of a minimum uv-cut.

u
v

Proposition
For u and v vertices of G, $\kappa(u,v) \geq \kappa(G)$.

Proof
A uv-cut makes the graph disconnected, so the connectivity is at most the size of a uv-cut.

Let λ(u,v) be the maximum number of internally disjoint uv-paths.

Proposition
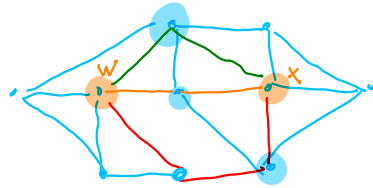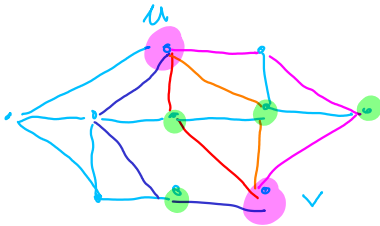For u and v vertices of G, κ(u,v) ≥ λ(u,v).

Proof
We need to delete at least one vertex per path, and no vertex belongs to two paths.

● Minimal uv-cut, size 4

● Minimal wx-cut, size 3

In fact, one can get a much stronger result:

Theorem (Menger, 1927)
If u and v are not adjacent, the minimum size of a uv-cut is the maximum number of internally disjoint uv-paths.

Proof (optional): read in the textbook, proof of theorem 4.2.17.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001.
Sections 4.1 and 4.2