Recall from the very first lecture the following problem:

Scheduling and avoiding conflicts
My high school used to have a very long exam sessions at the end of the year, and there were still some conflicts. I wish the administrators knew graph theory…
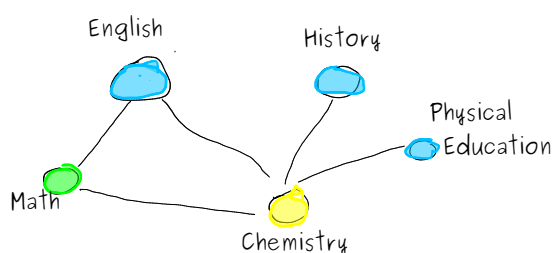
Schedule:
1. History—English—PE
2. Chemistry
3. Math

Vertices: Subjects
Edges: If someone takes both subjects, i.e. eventual scheduling conflicts.

Intuitive definition:
A coloring of a graph is a partition of a set into independent sets.

Scheduling
with no conflicts is equivalent to coloring. If we want to use the minimum time, we should use as few colors as possible.

Definition
A k-coloring of a graph G is a labeling of the vertices using labels from a set of size k (called colors, even though the labels can be numbers, for example).

The vertices of one color form a color class.

A coloring is proper if no two adjacent vertices have the same label.
A graph is k-colorable if it has a proper k-coloring.
The chromatic number χ(G) is the least k such that G is k-colorable.

In a proper coloring, every color class is an independent set. The chromatic number is the smallest number of independent sets in a graph.

Example

The Petersen graph has chromatic number 3:
- It is not 2-colorable, because its vertices cannot be divided into two independent sets; it would otherwise be bipartite.
- It is 3-colorable, as shown on the right.
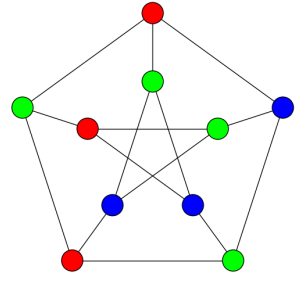
Notice that the chromatic number is an extremal problem: we need to show it is minimal and that a proper coloring exists.

Colorings for non-simple graphs

Graphs with loops do not admit proper colorings: a vertex that is incident to a loop could not be colored.

Every loopless graph can be colored: a trivial coloring where every vertex has a distinct color would work.

Multiple edges
Multiple edges don't change anything to colorings, as two adjacent vertices cannot be colored regardless of the number of edges between them.

Optimality

A graph G is k-chromatic if $k = \chi(G)$; a proper k-coloring is then an optimal coloring.
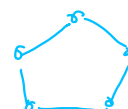
If $\chi(H) < \chi(G) = k$ for every subgraph H of G, then G is k-critical or color-critical.

Examples

k=1

k=2

k=3: The 3-critical graphs are the smallest graphs that are not bipartite: these are the odd cycles.

Not 3-critical

First bounds on the chromatic number
The clique number of a graph, written $\omega(G)$, is the maximum size of a clique in G. (Recall that a clique is a complete subgraph).

Also, recall that the independence number, $\alpha(G)$, is the size of a maximum independent set.
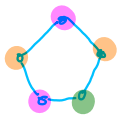
Proposition
For every graph G=(V,E), $\chi(G) \geq \omega(G)$ and $\chi(G) \geq |V|/\alpha(G)$.

Proof
If there is a clique of size k, the k vertices in the clique must be of different colors.
For the second inequality, rewrite it as $\chi(G)\alpha(G) \geq |V|$. $\chi(G)$ is the number of color classes, and $\alpha(G)$ is the maximum size of a color class.

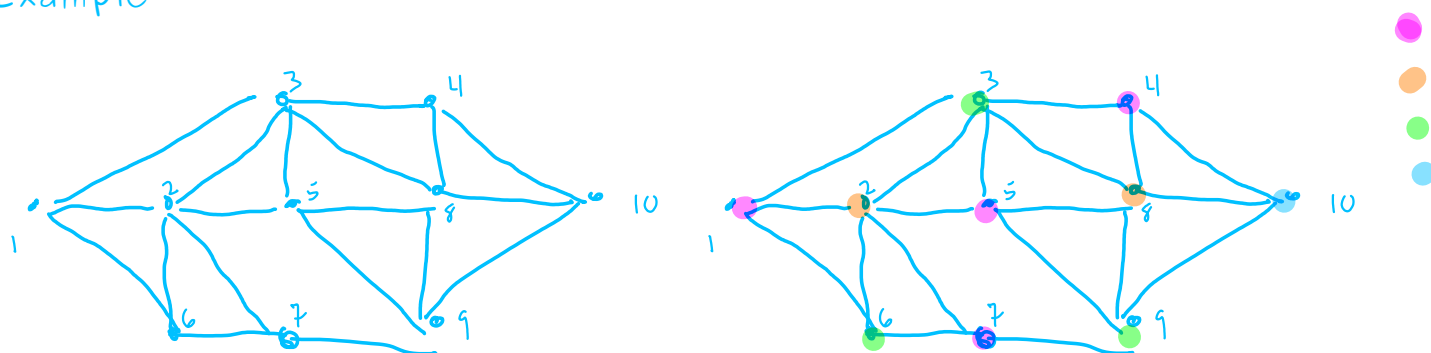The chromatic number is not necessarily the size of the maximal clique:

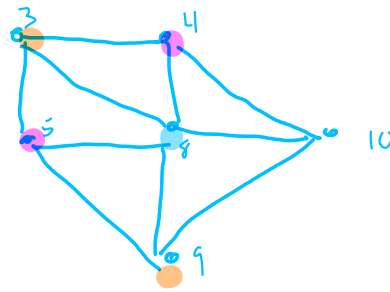Maximal clique has size 2
Chromatic number is 3

Greedy coloring algorithm

— Order the vertices {1,2,…,n}. We will color the vertices using numbers {1,2,…,n}.
— For every vertex (in order), label it with the smallest color not already in use in its neighborhood.
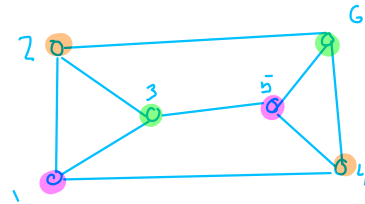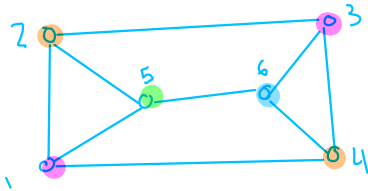
Example

In this case, it is actually minimal. This graph cannot be colored with fewer than 4 colors.

It is not always the case:

Proposition
The chromatic number is at most $\Delta(G)+1$.

Proof
The greedy algorithm described above yields a proper coloring. In the worst case, all neighbors of one vertex have distinct color, and we must add a color. When this happens, the number of colors is one more than the number of neighbors; that is at most $\Delta(G)+1$.

Reference: Douglas B. West. Introduction to graph theory, 2nd edition, 2001. Section 5.1.