Homework 2 - Solutions Spring 2020
Math 38 - Graph Theory

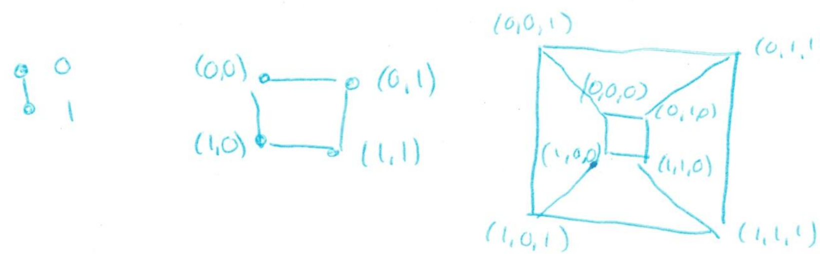
#2. All hypercubes are bipartite.

Proof:

We label the vertices of the hypercube as we construct them, in the following way:

- Label the vertices of H_1 0 and 1.
- H_{n+1} is made of two copies of H_n: In the first one, label the vertices like in H_n, and append a 0 at the end. In the second one, append a 1 at the end.

Example



We need to prove that this gives a bijection.

Let X be the set of vertices whose sum of coordinates is odd, and Y be its complement.

E.g., in H_3, $X = \{(0,0,1), (0,1,0), (1,0,0), (1,1,1)\}$.

I claim this is a bipartition since if u and v are adjacent in H_n, they differ by only one coordinate. We prove this by induction:

- In H_1, the two vertices differ only by one coordinate.
- In H_{n+1}, two vertices are adjacent if (1) they represent the same vertex in H_n or if (2) they are adjacent in H_n and are in the same copy.

  In (1), they differ by only the last coordinate.

  In (2), they share the last coordinate, which means, by induction hypothesis, they differ by only one coordinate.

Hence, all hypercubes are bipartite.

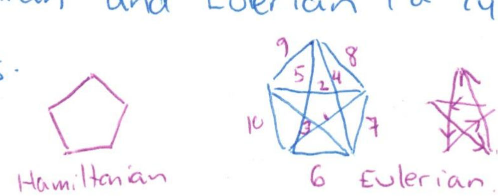
#3. The maximal independent sets of a bipartite and Hamiltonian graph have the same cardinality.

- Since it is Hamiltonian, it is connected. So there is only one bipartition into sets X and Y, or, the graph has only one vertex.
- Since it is Hamiltonian, the path has to visit all the vertices of the graph exactly once and end where it started.
- Because the graph is bipartite, any path alternates between vertices of X and of Y.
- If $|X| < |Y|$, then the path cannot visit all the vertices of Y and end where it started, so the graph is not Hamiltonian. The same holds if $|Y| < |X|$.

So $|X| = |Y|$, or there is only one set.

#4 (a) A trail that is both Hamiltonian and Eulerian is a cycle.
Let $G=(V,E)$ be a Hamiltonian graph. It has to be connected.
If a trail is Hamiltonian, it has length $|V|$, and a trail that is Eulerian has length $|E|$. Moreover, a Hamiltonian trail has no repeated vertex, so it is a path.
Since $|V| = |E|$, the whole graph is a cycle.

(b). A graph can be both Hamiltonian and Eulerian even if there does not exist one trail that is both Hamiltonian and Eulerian (a cycle, by part (a)). A counterexample is K_5.



#5. A graph whose vertices all have even degree has no cut-edge.

As seen in the lectures (cf. notes 4/05), a graph with vertices all of even degree can be decomposed into cycles.

However, an edge in a cycle cannot be a cut-edge (cf notes 4/06).

Since all edges belong to cycles, there is no cut-edge

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