Stability with Maximal Accuracy for a Hyperbolic Equation

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We describe a long-ago result for explicit difference methods of maximum accuracy for the hyperbolic advection equation $u_t = u_x$. Lax and Wendroff compute each value $U(t + \Delta t, n\Delta x)$ from three previous values $U(t, (n + k)\Delta x)$, $k = -1, 0, 1$. This has 2nd order accuracy and it is stable for $\Delta t \leq \Delta x$. The natural extension uses $2N + 1$ previous values $k = -N, \ldots, N$ and has order of accuracy $2N$.

Theorem : This extension is also stable for $\Delta t \leq \Delta x$ (but not all the way to the Courant-Friedrichs-Lewy limit $\Delta t \leq N\Delta x$).

Similarly, a 1st-order method uses $U(t, x)$ and $U(t, x + \Delta x)$ to compute $U(t + \Delta t, x)$. It is stable for $\Delta t \leq \Delta x$ (Courant-Friedrichs-Lewy condition). The natural "lopsided extension" with $k = 1 - N, \ldots, N$ uses $2N$ values of U at time t to compute each U at $t + \Delta t$. Its order of accuracy is $2N - 1$ and it is stable but only in the same range $\Delta t \leq \Delta x$. Both results extend to symmetric hyperbolic systems $u_t = Su_x$ (with symmetric matrix S).

For equally spaced interpolation $U(x)$ of $\exp{(ix)}$ at $2N + 1$ or $2N$ points, these results mean stability $|U(x)| \leq 1$ in the center interval (but not further).