On the real linear algebra of vectors of zeros and ones

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1 Prolog: Balanced and Unbalanced Collections

2 The matroid of 0-1 vectors
   - Unbalanced collections and the all-subset arrangement
   - Determinants of 0-1 matrices and Hadamard matrices
   - The characteristic polynomial and number of regions
   - Weak maps and bounds on the number of regions

3 Computing the characteristic polynomial
   - Finite field method and counting zeros mod $p$
   - Toward the broken circuit complex of $M_n$

4 Some Questions and References
For $S \subseteq [n] = \{1, 2, \ldots, n\}$, let $e_S := \sum_{i \in S} e_i$, where $e_i = (0, \ldots, 1, \ldots, 0)$ is the $i^{th}$ unit vector in $\mathbb{R}^n$. 
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A collection $\mathcal{F} \subseteq 2^{[n]}$ is said to be balanced if

$$\delta \cdot e_{[n]} \in \text{conv}\{e_S \mid S \in \mathcal{F}\}$$

for some $0 < \delta \leq 1$. 

Equivalently, $\mathcal{F}$ is balanced if the convex hull of the vertices of the cube $[0, 1]^n$ corresponding to the sets in $\mathcal{F}$ meets the diagonal.
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A collection is said to be unbalanced if it is not balanced.
Examples: balanced/unbalanced

\[
\{\{1, 2\}, \{1, 3\}, \{2, 3\}\} \text{ is balanced}
\]

\[
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We are interested in enumerating these collections.
Why care about such collections?

Balanced collections were introduced more than 50 years ago by Lloyd Shapley in his study of economic equilibria. Of particular interest were minimal balanced collections, which determine the minimum linear description of cooperative games possessing a nonempty "core".

Counting maximal unbalanced collections originally arose in thermal field theory = quantum field theory + statistical mechanics. Max'l unbalanced collections ↔ "Generalized Retarded Functions". This number has been computed through n=9:

2 3 4 5 6 7 8 9
2 6 32 370 11,292 1,066,044 347,326,352 419,172,756,930
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A few examples

For $n = 3$, the 6 maximal unbalanced collections are

\[ \{\{1, 2\}, \{1, 3\}, \{1\}\}, \{\{1, 2\}, \{2, 3\}, \{2\}\}, \{\{1, 3\}, \{2, 3\}, \{3\}\} \]

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e.g., for weight vectors $w = (2, -1, -1)$ and $w = (-2, 1, 1)$. 
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For $n = 4$, two of the 32 such collections are

$$\left\{ \{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\} \right\}$$

and

$$\left\{ \{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2\} \right\}$$
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Restricted all-subset arrangement in $\mathbb{R}^n$

Recall: $\mathcal{F} \subset 2^{[n]}$ is unbalanced $\iff$

$\exists w \in \mathbb{R}^n$, with $\sum_{i \in [n]} w_i = 0$ and $\sum_{i \in S} w_i > 0$ for $S \in \mathcal{F}$. 
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The \textbf{maximal (full-dimensional) regions} in this arrangement are in \textit{bijection} with the \textbf{maximal unbalanced collections} in $2^{[n]}$. 
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All-subset arrangement in $\mathbb{R}^{n-1}$

Combinatorially equivalent to the restricted all-subset arrangement in $\mathbb{R}^n$ is the all-subset arrangement $\mathcal{A}_{n-1}$ in $\mathbb{R}^{n-1}$, consisting of all hyperplanes with normals $e_S, S \subseteq [n-1], S \neq \emptyset$. Example: $n = 3$. The planes of $\mathcal{A}_2$ are $x_1 = 0$, $x_2 = 0$, $x_1 + x_2 = 0$, so $\mathcal{A}_2$ has 6 regions:
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$A_3$ has 7 planes and 32 regions
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The matroid of all 0-1 vectors

To count the regions of $A_n$, we need to understand the real linear matroid $M_n$ of all nonzero 0-1 vectors in $\mathbb{R}^n$. 

**Affine picture for $M_3$ a.k.a. non-Fano plane**

Note: two-point lines are not drawn.
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Determinants of 0-1 matrices

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$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -2.$$
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The matroid $M_n$ knows only when $\det A = 0/ \neq 0$. The arithmetic matroid of Moci, et al., knows also $|\det A|$. 

Hadamard (1893): For a 0-1 $n \times n$ matrix $A$, $|\det A| \leq (\frac{n+1}{2})^n$, with equality if and only if there exists a Hadamard matrix of order $n+1$. (So only if $n = 1$ or $n \equiv 3 \mod 4$.)
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How large can $|\det A|$ be for a 0-1 $n \times n$ matrix?

A Hadamard matrix is a $\pm 1$ $n \times n$ matrix whose rows (equiv. columns) are mutually orthogonal. These can only exist when $n = 1, 2$ or $4k$; they are conjectured to exist whenever $n = 4k$. 
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\[
\begin{vmatrix}
1 & 1 & 0 \\
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0 & 1 & 1
\end{vmatrix}
\]

\[\text{det} \begin{pmatrix}
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A Hadamard matrix is a \(\pm 1\) $n \times n$ matrix whose rows (equiv. columns) are mutually orthogonal. These can only exist when $n = 1, 2$ or $4k$; they are conjectured to exist whenever $n = 4k$. (The smallest unknown case is $n = 668 = 4 \cdot 167$.)
Determinants of 0-1 matrices

To “know” the matroid $M_n$ is to know about determinants of all 0-1 $n \times n$ matrices, for example

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -2.$$  

The matroid $M_n$ knows only when $\det A = 0/ \neq 0$. The arithmetic matroid of Moci, et al., knows also $|\det A|$.

How large can $|\det A|$ be for a 0-1 $n \times n$ matrix?

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Hadamard (1893): For a 0-1 $n \times n$ matrix $A$, $|\det A| \leq \frac{(n+1)^{\frac{n+1}{2}}}{2^n}$, with equality if and only if there exists a Hadamard matrix of order $n + 1$. (So only if $n = 1$ or $n \equiv 3 \mod 4$.)
Let $L_n = \text{lattice of flats of } M_n$, consisting of the linear (resp. affine) subspaces spanned by sets of 0-1 vectors, ordered by inclusion.
The characteristic polynomial of $M_n$

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The Möbius function of $L_n$ is defined for $x, y \in L_n, x \leq y$, by

$$\mu(x, x) = 1, \text{ and } \sum_{x \leq z \leq y} \mu(x, z) = 0 \text{ when } x < y.$$
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$$\chi(M_n, t) = \sum_{x \in L_n} \mu(0, x) \ t^{\text{rank}(L_n) - \text{rank}(x)} = \sum_{k=0}^{n} w_k(L_n) \ t^{n-k}$$
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This polynomial is known only through $n = 7$. 
\[\chi(M_1, t) = t - 1\]
\[\chi(M_2, t) = t^2 - 3t + 2 = (t - 1)(t - 2)\]
\[\chi(M_3, t) = t^3 - 7t^2 + 15t - 9 = (t - 1)(t^2 - 6t + 9)\]
\[\chi(M_4, t) = t^4 - 15t^3 + 80t^2 - 170t + 104 = (t - 1)(t^3 - 14t^2 + 66t - 104)\]
\[\chi(M_5, t) = t^5 - 31t^4 + 375t^3 - 2130t^2 + 5270t - 3485 = (t - 1)(t^4 - 30t^3 + 345t^2 - 1785t + 3485)\]
\[\chi(M_6, t) = t^6 - 63t^5 + 1652t^4 - 22435t^3 + 159460t^2 - 510524t + 371909 = (t - 1)(t^5 - 62t^4 + 1590t^3 - 20845t^2 + 138615t - 371909)\]
\[\chi(M_7, t) = t^7 - 127t^6 + 7035t^5 - 215439t^4 + 38318335t^3 - 37769977t^2 + 169824305t - 135677633 = (t - 1)(t^6 - 126t^5 + 6909t^4 - 208530t^3 + 3623305t^2 - 34146672t + 135677633)\]
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So, to determine the number of regions in $A_n = \text{the number of maximal unbalanced collections in } [n + 1]$, we could try to determine the characteristic polynomial $\chi(M_n, t)$. 
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So, to determine the number of regions in \( A_n \) = the number of maximal unbalanced collections in \([n + 1] \), we could try to determine the characteristic polynomial \( \chi(M_n, t) \).

We first get an easy lower bound on this number.
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$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \equiv 0 \mod 2$$
The “binary all-subsets arrangement”

Consider the binary matroid $M_n^2$ generated over the 2-element field $\mathbb{F}_2$ by all the nonzero elements of $\{0, 1\}^n$, i.e., the projective geometry of rank $n$ (dimension $n - 1$) over $\mathbb{F}_2$, $PG(n - 1, 2)$. 

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Theorem: The number of maximal unbalanced families in \([n + 1]\), equivalently, the number of chambers of the arrangement \(A_n\), is at least \(\prod_{i=0}^{n-1}(2^i + 1)\). Thus the number of maximal unbalanced collections is more than

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Note: Zuev (1989) had effectively shown it is asymptotically \(2^{n^2}\). His argument uses a theorem of Odlyzko on random \(\pm 1\) vectors.
Computing $\chi(\mathcal{H}, t)$ – the finite field method

For polynomial $f = f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$, define for a prime $p$

$$N_f(p) := \left| \left\{ (r_1, \ldots, r_n) \in (\mathbb{F}_p)^n \mid f(r_1, \ldots, r_n) = 0 \text{ in } \mathbb{F}_p \right\} \right|$$
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So for $p$ large enough, $N_{f_{\mathcal{H}}}(p)$ is a polynomial in $p$ (positive lead term, alternating signs), i.e., $\mathcal{H}$ is a "polynomial count variety". As are the Grassmannian and the flag variety. Are there others?
To compute $\chi(M_n, t)$ we need to compute $N_{f_n}(p)$, where

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For example, $f_3 = x_1x_2x_3(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)(x_1 + x_2 + x_3)$, and $N_{f_3}(p) = 7p^2 - 15p + 9$ when $p > 2$, since we know

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General problem: Count the zeros mod $p$ of your favorite symmetric and other combinatorially defined polynomials.
For a matroid $M$ on a linearly ordered set $X$, a
- **circuit** is a minimally dependent set
- **broken circuit** is a circuit minus its largest element
Computing $\chi(M, t) – the broken circuit complex$

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\[ BC(M) := \{ \sigma \subset X \mid \sigma \text{ contains no broken circuit} \} \]
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Theorem (Whitney, 1932): $|w_k(M)| = f_{k-1}(BC(M))$
The reduced broken circuit complex

Note that if $\bar{x}$ is the largest element in $X$, then $\bar{x}$ is in no broken circuit, so $BC(M)$ is a cone with apex $\bar{x}$. 

$\chi(M, t) = t^3 - 7t^2 + 15t - 9 = (t-1)(t^2 - 6t + 9)$

$BC(M) = (1,1,1) \ast BC(M)$

$f(BC) = (f-1, f_0, f_1) = (1, 6, 9)$

$f(BC) = (f-1, f_0, f_1, f_2) = (1, 7, 15, 9)$
The reduced broken circuit complex

Note that if $\bar{x}$ is the largest element in $X$, then $\bar{x}$ is in no broken circuit, so $BC(M)$ is a cone with apex $\bar{x}$. Define the reduced broken circuit complex to be

$$\overline{BC}(M) := BC(M) \setminus \bar{x}$$

so that $BC(M) = \bar{x} \ast \overline{BC}(M)$
The reduced broken circuit complex

Note that if $\bar{x}$ is the largest element in $X$, then $\bar{x}$ is in no broken circuit, so $BC(M)$ is a cone with apex $\bar{x}$. Define the reduced broken circuit complex to be

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Order $\{0, 1\}^n$ lexicographically (as if base 2 numbers) and recall

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$BC(M_3)$

![Diagram of the reduced broken circuit complex for $M_3$.]
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f(BC) = (f_{-1}, f_0, f_1, f_2) = (1, 7, 15, 9)
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What can we say about $f_i(BC(M_n))$?

Unfortunately, not very much.

$$f_{−1} = 1$$

$$f_0 = 2$$

$$n−1$$

$$f_1 =$$

$$n−3$$

$$n−2$$

$$2$$

$$= 3$$

$$S((n+1),4) + 2S((n+1),3)$$

... Sterling #'s

The last is new. For example,

$$(4^3−3^3−2^3+1)/2 = (64−27−8+1)/2 = 15 = f_1(BC(M_3))$$.

To understand $f_1(BC(M_n))$, we must understand lines in the affine picture for $M_n$.

Recall the identity map $M_n \rightarrow M_{2n}$ is a rank-preserving weak map (inverse image of independent sets are independent).

$M_{2n}$ is a binary matroid, so we'll call the simple matroid $M_n$ "weakly binary" since it has a simple binary matroid as bijective weak image.

Proposition: k-flats in weakly binary matroids have at most $2^{k−1}$ points, so lines have either 2 or 3 points.
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Broken circuits on lines

In a weakly binary matroid,

- the only pairs that can contain broken circuits are broken circuits, and
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\[ f_1(BC(M_n)) = \binom{2^n - 1}{2} - \text{the number of 3-point lines} \]

A “line” in \( M_n \) is spanned by two distinct points \( e_S \) and \( e_T \), \( S, T \subset [n] \).
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So the number of broken circuits of size 2 is precisely the number of disjoint pairs of sets in \( [n] \), i.e., \( \frac{3^n-2^{n+1}+1}{2} \). \( \square \)
Some questions

Determine $\chi(M_n, t)$ exactly for all $n$. Kamiya, Takemura and Terao have computed it for $n \leq 8$.

More specifically, determine $f_i(BC(M_n))$, for $n \geq 8$ and $i \geq 2$. There is some hope that $i = 2$ is a key here (cf. Odlyzko).

Are there other combinatorially interesting polynomial count varieties (besides rational hyperplane arrangements, Grassmannians and flag varieties)?

Count the zeros mod $p$ of your favorite symmetric and other combinatorially defined polynomials, e.g. $s_\lambda$.

In general, can algebraic combinatorics be useful to arithmetic geometry?

The set of all maximal unbalanced collections in $[n]$ forms a pure simplicial complex of dimension $n - 1$. What is its topology? For $n = 3$, it is $S_1 \times I$.

Minimal balanced collections are broken circuits. They also can be viewed as generalized partitions. Is there a nice poset structure for them, say, ordered by "balanced refinement"?

(Partition lattice = lattice of flats of graphic matroid of $K_n$.)

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References (on counting $p$-points)

*Lectures on $N_X(p)$*
J.P. Serre

*Hyperplane Arrangements: An Introduction*
Alexandru Dimca
Lectures on $N_X(p)$
J.P. Serre

Hyperplane Arrangements
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[includes references to the economics/physics applications, in particular:]


Thank you!!