Exponents of Primitive Digraphs

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1 Introduction

Let A be an $n \times n$ matrix whose entries are nonnegative and let $D = (V, E)$ be the digraph associated with A where $V = [n]$ and $(i, j) \in E$ if $a_{i,j} > 0$. We call A *primitive*, if there exists some k such that A^k has positive entries. Equivalently, we call D *primitive*, if there exists a positive integer k such that there is a directed walk between any pair of vertices in D of length k (number of edges in the walk). Let D^k be the digraph that has the same vertices as D where there is a directed edges between two vertices u and v if there is a directed walk of length k from v to u. The smallest k satisfying this property is the *exponent* of D and is denoted by $\gamma(D)$.

2 Main Theorem

Theorem 1. *If D is a primitive digraph, then $\gamma(D) \leq n^2 - 2n + 2$.*

In order to prove this theorem, we will show a lemma due to [4]. Let us say that a walk *touches* a given set of vertices if there is some vertex in that set which belongs to the walk.

Lemma 2. *If D be a primitive digraph with n vertices, let D' be a subset containing n_s distinct vertices of the digraph, and let v be any vertex in D. Then there is always a walk from v which touches D' whose length is less than or equal to $n - n_s$.*

Proof. If $v \in D'$, then the lemma is trivial. Suppose $v \in D'$. By primitivity, we know that there is at least one walk which starts at v and touches D'. Let W be such a walk of shortest length and let $u \in D'$ be the vertex where this walk ends. This walk has no repeated vertices and u is the only vertex in D' that occurs in W. Hence, W has at most $n - n_s$ vertices. □

Proof. (**Theorem 1**)[P. Winkler] We know that there is a cycle C in D of length $k < n$; if not, then D is a single cycle which contradicts the primitive property of D. We also know from Lemma 2 that from any vertex v there is a walk of length at most $n - k$ to C. We can get from C to any vertex u in exactly $k(n-1)$ steps: in D^k, due to the loops at vertices in C, we can get to u in $m = n - 1$ steps; therefore, in D we can get to u in $k(n-1)$ steps. Thus, we can get from v to u in $n - k + k(n-1)$ steps. For $k = (n-1)$, we have $n - k + k(n-1) = 1 + (n-1)^2$. Now for $k \leq (n-1)$, we have $n - (k-1) + (k-1)(n-1) = n - k + k(n-1) + 1 - (n-1) = n - k + k(n-1) - (n-2)$. Since $n \geq 2$, then for $k < (n-1)$, $n - k + k(n-1) \leq 1 + (n-1)^2$. □

Now we will show some examples that satisfy this upper bound.

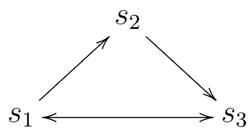

Figure 1: Δ

Example 3. *Let Δ be the digraph depicted in Figure 1. For this digraph, we have $\gamma(\Delta) = 5 = (3-1)^2 + 1$: let A be the associated matrix to Δ and we have*

$$A = \begin{pmatrix} 0 & + & + \\ 0 & 0 & + \\ + & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} + & 0 & + \\ + & 0 & 0 \\ 0 & + & + \end{pmatrix}$$

$$A^4 = \begin{pmatrix} + & + & + \\ + & 0 & + \\ + & + & + \end{pmatrix}$$

where $+$ represents a positive entry.

3 A Concise History of the Upper Bounds for $\gamma(D)$

In 1950, H. Wielandt stated without proof that $\gamma(A) \leq n^2 - 2n + 2$ for a primitive $n \times n$ matrix. In the special case that all the diagonal entries of A are positive, he showed that $\gamma(A) \leq n - 1$. P. Perkins proved Theorem 1 in 1961. In 1964, A.L. Dulmage and N.S. Mendelsohn proved the following which was also proved by E.V. Denardo in 1977:

Theorem 4. *If D is a primitive digraph on n vertices and if s is length of the shortest cycle in D, then $\gamma(D) \leq n + s(n-2)$.*

4 Minimally Strong, Primitive Digraphs

A digraph D is called *strongly connected* (*strong*) if there is a walk from any vertex v to any vertex $u \neq v$ in D. Moreover, a strong graph is called *minimally strong* given that removal of an edge will result in a digraph that is not strong.

Theorem 5 (R. Brualdi, J. Ross – 1980)**.** *Let D be a primitive, minimally strong digraph on n vertices. Then*

$$\gamma(D) \leq n^2 - 4n + 6,$$

with equality if and only if D is isomorphic to the digraph D_n.

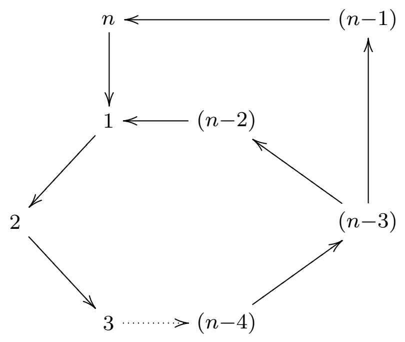

Figure 2: D_n

Let D be a primitive digraph and let $p_1, p_2, \ldots, p_k$ be the distinct lengths of closed path of D. Define $r_{u,v}$ to be the length of the shortest walk from vertex u to vertex v such that for all $i \in [k]$, it contains a vertex of some closed path of length p_i. Let $r(D) = \max\{r_{u,v}\}$. An ordered pair u, v has the *unique path property* if every walk from u to v of length at least $r_{u,v}$ consists of some walk W from u to v of length $r_{x,y}$ augmented by closed paths each of which has a vertex in common with W.

Let $n_1, n_2, \ldots, n_k$ be relatively prime positive integers, and let $F(n_1, n_2, \ldots, n_k)$ be the largest integer that is not expressible as linear composition of n_i's with nonnegative coefficients. Dulmage and Mendelsohn proved in [2] the following lemma:

Lemma 6. *Let D be a primitive digraph for which p_1,$p_2, \ldots$,p_k are the distinct lengths of closed paths. Then*

$$\gamma(D) \leq F(n_1, n_2, \ldots, n_k) + r(D) + 1. \tag{1}$$

If the pair u, v has the unique path property, then

$$F(p_1, p_2, \ldots, p_k) + r_{u,v} + 1 \leq \gamma(D). \tag{2}$$

Proof. (**Theorem 5**, Outline) Clearly, $r(D_n) = r_{n-1,n} = n$. We use (1) and $F(n,m) = nm - n - m$ to show that $\gamma(D_n) \leq n^2 - 4n + 6$. Since the ordered pair $n-1, n$ has the unique path property, then using (2), we have $n^2 - 4n + 6 \leq \gamma(D_n)$.

Let D be a minimally strong, primitive digraph. We know that D has a closed path of length $k < n$, and let s be the minimum length of a closed path. The case $s = n-1$ can not happen. Hence, $s \leq n-2$. When $s \leq n-4$, we know from **Theorem 4** that

$$\gamma(D) \leq n^2 - 5n + 8 < n^2 - 4n + 6.$$

When $s = n-2$, D has a closed path of length $n-1$. Using some basic properties of digraphs, it is easy to show that D is isomorphic to D_n. When $n = s-3$, D has a closed path C_1 of length $n-3$. It also has a closed path C_2 of length $l = n-1$ or $l = n-2$. When $l = n-2$, two or three vertices of C_2 are not in C_1. It follows that the digraph D' induced on the vertices of C_1 and C_2 is isomorphic to D_{n-1} or the digraph E_n in Figure 3. In either case, $\gamma(D) < n^2 - 4n + 6$. When $l = n-1$, using basic properties of digraphs, we can show that D is the digraph F_n in Figure 4. Using a similar argument as was given for D_n, it can be proved that $\gamma(F_n) < n^2 - 4n + 6$. $\square$

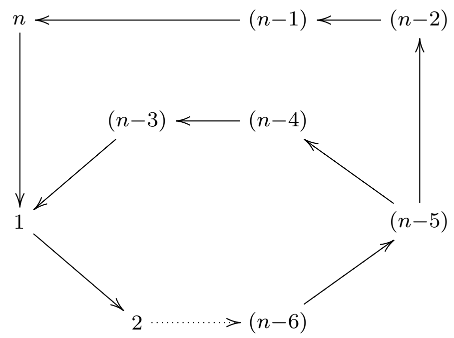

Figure 3: E_n

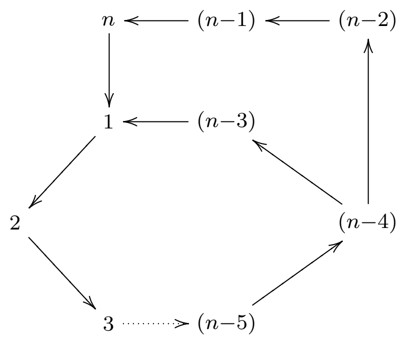

Figure 4: F_n

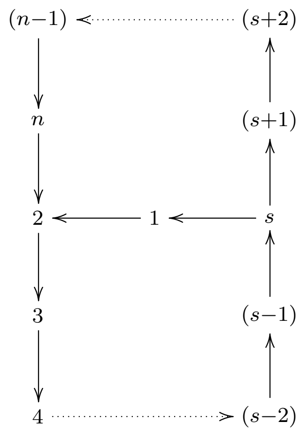

Figure 5: $D_{s,n}$

We will finish this project paper by stating the following theorem that improves the bound in **Theorem 4** when D is a minimally strong, primitive digraph:

Theorem 7 (J. Ross – 1982)**.** *Let D be a minimally strong, primitive digraph on n vertices. Then*

$$\gamma(D) \leq n + s(n-3)$$

with equality if and only if D is isomorphic to the digraph $D_{s,n}$. In particular, if $g.c.d(s, n-1) \neq 1$, then $\gamma(D) < n + s(n-3)$, and if $g.c.d(s, n-1) = 1$, then $D_{s,n}$ is a primitive, minimally strong digraph on n vertices with exponent $n + s(n-3)$.

References

[1] E.V. Denardo, *Periods of connected networks and powers of nonnegative matrices*, Math. Oper. Res., 2 (1977), pp. 20–24.

[2] A.L. Dulmage, N.S. Mendelsohn, *Gaps in the exponent set of primitive matrices*, Illinois J. Math., 8 (1964), pp. 642-656.

[3] A.L. Dulmage, N.S. Mendelsohn, *The exponent of a primitive matrix*, Canad. Math. Bull., vol. 5 (1962), pp. 241–244.

[4] P. Perkins, *A theorem on regular graphs*, Pacific J. Math., vol. II (1961), pp. 1529–1533.

[5] H. Wielandt, *Unzerlegbare, nicht negativen Matrizen*, Math. Zeitschrift, vol. 52 (1950), pp. 642–648.